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Cohomology and other analytical aspects of RCD spaces

Ph.D. Thesis

Supervisor
Prof. Nicola Gigli

Candidate
Chiara Rigoni

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SISSA - Via Bonomea 265 - 34136 TRIESTE - ITALY

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Abstract

This thesis is primarily devoted to the study of analytic and geometric properties of metric measure spaces with a Ricci curvature bounded from below.

The first result concerns the study of how a hypothesis on the Hodge cohomology affects the rigidity of a metric measure space with non negative Ricci curvature and finite dimension: we prove that if the dimension of the first cohomology group of a $\mathsf{RCD}^*(0, N)$ space is N, then the space is a flat torus. This generalizes a classical result in Riemannian geometry due to Bochner to the non-smooth setting of RCD spaces.

The second result provides a direct proof of the strong maximum principle on finite dimensional RCD spaces mainly based on the Laplacian comparison of the squared distance.

Ai miei genitori, a Trieste e
a tutti coloro che hanno reso speciali questi quattro anni.

Contents

Introduction

In the last years the study of metric measure spaces with a lower Ricci curvature bound has led to many interesting results on the analytic and geometric properties of these spaces (we refer for example to [70], [71], and the references therein for a complete overview of the topic). These include diameter bounds, volume comparison theorems, heat-kernel and spectral estimates, Harnack inequalities, topological implications, Brunn-Minkowski-type inequalities, and isoperimetric, functional and concentration inequalities (see for instance [9], [21], [52], and [69]). Actually there are also many rigidity results which have been recently generalized in the setting of $\mathsf{RCD}^*(K,N)$ spaces, for example:

- the splitting theorem in $\mathsf{CD}(0,N)$ spaces (see [35], [37]), where Gigli proves that an infinitesimally Hilbertian $\mathsf{CD}(0,N)$ space containing a line splits as the product of $\mathbb{R}$ and an infinitesimally Hilbertian $\mathsf{CD}(0,N-1)$ space;
- the maximal diameter theorem, proved by Ketterer in [50], that has been obtained as a corollary of the splitting theorem together with the other result of the paper, which asserts that the cone over some metric measure space satisfies the reduced Riemannian curvature-dimension condition $\mathsf{RCD}^*(KN,N+1)$ if and only if the underlying space satisfies $\mathsf{RCD}^*(N-1,N)$;
- the validity on RCD spaces of a result by Cheeger-Colding, valid in Ricci-limit spaces, which says that volume cone implies the metric cone, proved by De Philippis and Gigli in [28];
- Obata's rigidity theorem for metric measure spaces that satisfy a Riemannian curvature-dimension condition, proved by Ketterer in [51];
- the study of the case of the equality in the Levy-Gromov inequality, due to Cavalletti and Mondino in [26].

The main core of my thesis consists in the generalization to the non smooth setting of RCD spaces of a result, due to Bochner, valid in Riemannian geometry. In particular this result relies the study of how a hypothesis on the Hodge cohomology affects the rigidity of a metric measure space with non negative Ricci curvature and finite dimension. In the proof of this result a key role is played by the theory of Regular Lagrangian flows on metric measure spaces.

Furthermore, in the last chapter of this thesis we show a direct proof of the strong maximum principle on finite dimensional RCD spaces, which is mainly based on the Laplacian comparison of the squared distance.

Weak bounds on Ricci curvature. The Curvature-Dimension condition $\mathsf{CD}(K,N)$ was first introduced by Bakry and Émery in the 1980's in the context of diffusion generators (see [20]). In this case the setting is the one of weighted Riemannian manifolds, that are smooth Riemannian manifolds equipped with a smooth density with respect to the volume measure: this condition provides a generalization in the non-weighted Riemannian manifolds of the classical notion of having Ricci curvature bounded from below by $K \in \mathbb{R}$ and dimension bounded from above by $N \in [1,\infty]$.

However this definition involves the differential calculus on the manifold as well as the Riemannian structure of the space, therefore at that time it was not clear how to extend it beyond the smooth Riemannian setting, for example to generic metric measure spaces appearing as (measured) Gromov-Hausdorff limits of a sequence of Riemannian manifolds.

The key ingredient that allowed the theory to proceed in this direction comes from the framework of Optimal Transport (we refer for example to [69]). Indeed, since for every couple of probability measures μ_0, μ_1 on a common geodesic metric space (M, d) there exists a Wasserstein geodesic $[0,1] \ni t \mapsto \mu_t$ connecting μ_0 and μ_1 in the space of probability measures $\mathscr{P}(\mathrm{M})$, we can introduce the notion of displacement convexity of a given functional on $\mathscr{P}(\mathrm{M})$ along Wasserstein geodesics (see [54]). Then, from a work of Sturm and Von Renesse [72], it turns out that the $\mathsf{CD}(K,\infty)$ condition (namely the fact that the Ricci curvature is bounded from below by a constant $K \in \mathbb{R}$) for a smooth manifold can be equivalently reformulated synthetically as a suitable convexity property of an entropy functional along Wasserstein geodesics (associated to L^2-Optimal Transport when the transport-cost is the squared-distance function).

This result allows to generalize the $\mathsf{CD}(K,N)$ condition for a generic metric measure space: indeed, separately, Lott and Villani in [53], and Sturm in [67] and [68], introduce the synthetic definition of $\mathsf{CD}(K,N)$ on a general complete and separable metric space (M, d), equipped with a Radon reference measure $\mathfrak{m}$. This notion coincides with the one proposed by Bakry-Émery in the smooth Riemannian setting (as proved by Ambrosio, Gigli, and Savaré in [10]), and, in particular, also Finsler manifolds and Alexandrov spaces satisfy the Curvature-Dimension condition (we refer respectively to the work of Ohta [56] and Petrunin [59]). Moreover this notion is stable under measured Gromov-Hausdorff convergence of metric measure spaces and, in analogy with the smooth setting, many geometric and analytic inequalities relating metric and measure can be proved.

The idea behind the $\mathsf{CD}(K,N)$ condition is to prescribe a synthetic bound on how an infinitesimal volume changes when it moves along a W^2-geodesic, namely how it is affected by curvature when it is moved via optimal transportation. Condition $\mathsf{CD}(K,N)$ imposes that the distortion is ruled by the coefficients $\tau_{K,N}^{(t)}(\theta)$ depending on the curvature K, on the dimension N, on the time of the evolution t, and on the point θ. Hence for a general (geodesic) metric measure space $(\mathrm{M}, \mathsf{d}, \mathfrak{m})$ this notion of lower bound on the Ricci curvature and upper bound on the dimension allows to study the geometry of these spaces by means of optimal transport.

The lack of the local-to-global property of the $\mathsf{CD}(K,N)$ condition (for $K/N \neq 0$) led Bacher and Sturm to introduce the so called reduced curvature-dimension condition, denoted by $\mathsf{CD}^*(K,N)$ (we refer to [17]). This condition require the same convexity property along Wasserstein geodesics of the entropy functional considered to give the definition of $\mathsf{CD}(K,N)$, but the distortion coefficients $\tau_{K,N}^{(t)}(\theta)$ are now replaced by slightly smaller ones.

Without requiring any non-branching assumptions, the $\mathsf{CD}^*(K,N)$ condition implies the same geometric and analytic inequalities as the $\mathsf{CD}(K,N)$ one, but with slightly worse constants. Indeed this condition is a priori weaker than the $\mathsf{CD}(K,N)$ and a converse implication can easily be obtained only changing the value of the lower bound on the curvature: condition $\mathsf{CD}^*(K,N)$

implies $\mathsf{CD}(K^*, N)$, where $K^* = K(N-1)/N$.

A natural step forward in this theory was made introducing the Riemannian curvature dimension condition $\mathsf{RCD}^*(K, N)$: in the infinite dimensional case the definition of $\mathsf{RCD}(K, \infty)$ was given by Ambrosio, Gigli, and Savaré in [9] for finite measures $\mathfrak{m}$ and by Ambrosio, Gigli, Mondino, and Rajala in [5] in the case of σ-finite measures. As for the class of $\mathsf{RCD}^*(K, N)$ spaces with $N < \infty$ we refer to the work of Gigli [38], where this notion was first proposed, and to [14] and [31] for a further investigation of these spaces. This condition is a straightening of the reduced curvature-dimension one: indeed a metric measure space is $\mathsf{RCD}^*(K, N)$ if and only if it satisfies $\mathsf{CD}^*(K, N)$ and is infinitesimally Hilbertian, meaning that the Sobolev space is a Hilbert space (where the Hilbert structure is induced by the Cheeger energy).

It is worth to recall that in the recent paper [25], Cavalletti and Milman proved the equivalence of $\mathsf{CD}(K, N)$ and $\mathsf{CD}^*(K, N)$ conditions, as well as the local-to-global property for $\mathsf{CD}(K, N)$ spaces, provided the metric measure space $(\mathrm{M}, \mathsf{d}, \mathfrak{m})$ is essentially non-branching and the $\mathfrak{m}$-measure of the whole space is finite.

Actually in [61], Rajala and Sturm proved that a metric measure space verifying $\mathsf{RCD}^*(K, N)$ is essentially non-branching, implying also the equivalence of $\mathsf{RCD}^*(K, N)$ and $\mathsf{RCD}(K, N)$.

Differential calculus on RCD spaces. Alexandrov and RCD spaces are structures carrying many geometric information, therefore a natural question is whether we can generalize in this non smooth setting all the differential objects proper of Riemannian manifolds.

In the context of Alexandrov spaces with a bound from below on the curvature, the construction of a differential structure is mainly based on the concavity properties of the distance function, which are part of the definition of the spaces themselves. This regularity information is enough in order to state the calculus rules needed to create a non-trivial second order calculus (we refer to [1] for an overview on the topic).

As for the more general setting of metric measure spaces, it has been proved that a first order differential structure always exists (we refer to the work of Cheeger [27] and to the ones of Weaver [73] and [74]), while a second order one arises when a lower Ricci bound is imposed (see [36] and the related work [33]). The approach followed in the construction is analytic, in the sense that provides all the tools needed to make computations on metric measure spaces, without having a priori relation with their geometry. For example the definition of 'tangent space' introduced in this way is not the one given in terms of pointed-measured-Gromov-Hausdorff limit of rescaled spaces, and actually the two notions can be very different when considering irregular spaces.

The key notion necessary in order to define the cotangent module $L^2(T^*\mathrm{M})$ is the one of L^2-normed L^∞-module, which provides a suitable abstraction of the notion of 'space of L^2-sections of a vector bundle'. Briefly, an L^∞-module is a Banach space whose elements can be multiplied by functions in $L^\infty(\mathfrak{m})$, while an L^2-normed L^∞-module is an L^∞-module for which there exists a function, called pointwise norm, which associates to every element of the module a non-negative function in $L^2(\mathfrak{m})$ such that its $L^2(\mathfrak{m})$-norm is compatible with the norm on the Banach space. Indeed, the analogy with the smooth case is given by the fact that the space of smooth sections of a vector bundle on a smooth manifold M can be described via its structure as module over the space $C^\infty(M)$ of smooth functions on M. Now, replacing the smoothness assumption with the integrability one, we find that the space of L^2-sections of a normal bundle on M can be described as a module over the space of L^∞ functions on M. Actually the notion of L^2-normed L^∞-module has the advantage to revert the procedure described just before, allowing to speak about L^2 sections of a vector bundle, without really having the bundle.

Adopting this point of view, we declare that tensor fields on a metric measure space are $L^\infty(\mathfrak{m})$-modules. We remark that this means that tensors will never be defined pointwise, but only given $\mathfrak{m}$-almost everywhere. In particular we will say that a (co)tangent vector field is an element of the (co)tangent module.

The idea of using this notion of $L^\infty(\mathfrak{m})$-modules to provide an abstract definition of vector fields in the non-smooth setting has been proposed by Weaver in [74], who was in turn inspired by the papers [62], [63] of Sauvageot. The biggest difference between the two approaches is due to the fact that in [36] all the constructions are based on Sobolev functions, while in [74] on Lipschitz ones.

We fix a metric measure space $(\mathrm{M}, \mathsf{d}, \mathfrak{m})$ which is complete and separable as a metric space, and it is equipped with a non-negative Radon measure $\mathfrak{m}$. As for the first order differential structure, we begin the construction introducing the class of Sobolev functions $W^{1,2}(\mathrm{M})$, which are functions in $L^2(\mathfrak{m})$ for which it is well defined the notion of minimal upper gradient. Moreover for a given Sobolev function we can define its differential, which in particular is an element of the cotangent module $L^2(T^*\mathrm{M})$.

Once we have the notion of cotangent module, the tangent module $L^2(T\mathrm{M})$ can be introduced by duality as the space of linear continuous maps $L\colon L^2(T^*\mathrm{M}) \to L^1(\mathfrak{m})$ satisfying

$$L(f\omega) = fL(\omega), \qquad \forall\, \omega \in L^2(T^*\mathrm{M}),\ f \in L^\infty(\mathfrak{m}).$$

This space carries a natural structure of $L^2(\mathfrak{m})$-normed module, so in particular for every element $X \in L^2(T\mathrm{M})$, that we call vector field, the pointwise norm of $|X|$ is a well defined $L^2(\mathfrak{m})$ function.

At this point we have the general first order theory and so we can move to the study of the second order differential structures of $\mathsf{RCD}(K,\infty)$ spaces. The main result of this construction is the improvement of the Bochner inequality, from

$$\Delta\frac{|\nabla f|^2}{2} \geq \langle \nabla f, \nabla \Delta f\rangle + K|\nabla f|^2, \tag{0.1}$$

where f is an element of a sufficiently large class of functions, to the more general one

$$\Delta\frac{|X|^2}{2} \geq |\nabla X|^2_{HS} - \langle X, (\Delta_H X^\flat)^\sharp\rangle + K|X|^2, \tag{0.2}$$

to be understood in the appropriate weak sense. We remark that in the case in which $X = \nabla f$, (0.2) reduces to (0.1) with the additional non negative contribution of $|\mathrm{Hess} f|^2_{HS}$ in the right-hand side. In particular the language of $L^2(\mathfrak{m})$-normed modules gives rise to spaces where objects like the Hessian of a function or the covariant derivative of a vector field belong to, and one of the consequences of formula (0.2) is the following bound:

$$\int |\mathrm{Hess}(f)|^2_{\mathrm{HS}}\,\mathrm{d}\mathfrak{m} \leq \int (\Delta f)^2 - K|\nabla f|^2\,\mathrm{d}\mathfrak{m}, \tag{0.3}$$

that we can obtain integrating (0.2) for $X = \nabla f$. Since the heat flow allows to construct functions with gradient and Laplacian in L^2, the bound in (0.3) ensures that there are ‘many’ functions with Hessian in L^2. This is a very important point since it allows to built a second order calculus, which in turns provides a natural extension of the De Rham cohomology and of the Hodge theorem.

Furthermore, from the fact that we have at our disposal well-defined differential operators, we can define the Ricci curvature $\mathrm{Ric}(X,X)$ as the measure-valued tensor for which the Bochner identity holds:

$$\mathrm{Ric}(X,X) := \Delta\frac{|X|^2}{2} - |\nabla X|^2_{\mathrm{HS}} + \langle X, (\Delta_H X^\flat)^\sharp\rangle. \tag{0.4}$$

In particular (0.2) guarantees that in a $\mathsf{RCD}(K,\infty)$ space the Ricci curvature is bounded from below by K.

Finally we remark that the differential operators introduced in this way are stable with respect to measured-Gromov-Hausdorff convergence (we refer to [11], [12], and [46] for a detailed discussion on this topic).

Regular Lagrangian flows. In order to describe the origins of this theory, let us consider the system of ODE's:

$$\begin{cases} x'(t) = X(t,x(t)), & t \in (0,T) \\ x(0) = x, \end{cases} \tag{0.5}$$

where $X(t,x) = X_t(x)$ is a (possibly) time-dependent family of vector fields in $\mathbb{R}^n$. A natural question is whether we can give a meaning to this ODE in the case in which the vector fields X_t are not smooth or they are defined up to Lebesgue negligible sets. Indeed this problem arises in fluid mechanics (where the velocity typically belongs to a Sobolev class) and in the theory of conservation laws (where the velocity can be BV). We refer to [2] and to [4] for the developments in this direction.

The classical well-posedness theory of (0.5) requires the regularity on the vector field given by

$$\mathrm{Lip}(X_t) \in L^1(0,T)$$

and it ensures Lipschitz regularity with respect to x of the flow map $\mathrm{Fl}(t,x)$, as well as stability with respect to approximations of the vector field.

We remark that pointwise stability and uniqueness are still valid for vector fields with a special structure, for example satisfying a one-sided Lipschitz condition

$$\langle X_t(x) - X_t(y), x-y\rangle \leq C(t)|x-y|^2,$$

where $C \in L^1(0,T)$, or autonomous and gradient vector fields $X = -\nabla V$, with V convex. More in general, a local Lipschitz condition ensures that there exists a unique maximal flow, with a lower semicontinuous maximal existence time $T_X : \mathbb{R}^n \to (0,T]$.

In [30] Di Perna-Lions first introduced an appropriate notion of 'almost everywhere well-posedness' suitable for a large class of Sobolev vector fields. In particular the approach used in the construction of this theory is based on the method of characteristics and on the transport equation. Then in [2] Ambrosio revisited and improved the axiomatization of [30] in order to extend the theory to the case of BV vector fields. This has been done by introducing a new more probabilistic method, based on the duality between flows and continuity equation.

The classical non-uniqueness example on the real line, that provides an illustration of the kind of phenomena that can occur and explains this new point of view, is the following one: we consider the ODE

$$x' = \sqrt{|x|}, \quad x_0 = -c^2, \quad c \geq 0.$$

In this case we have $x(t) = -(t/2-c)^2$ for $0 \leq t \leq 2c$, which means that the solution can stay at the origin for some time $2T(c)$ (that can be also infinite or null), and then continue as $x(t) = (t/2 - T(c) - c)^2$ for $t \geq 2c + 2T(c)$, if $T(c) < \infty$. Therefore, for any measurable choice of $T(c)$, we have the flow map

$$\mathrm{Fl}(t,x)\colon [0,\infty) \times (-\infty,0] \to \mathbb{R}.$$

However it is possible to show that if we approximate the vector field X with different Lipschitz approximations, they produce in the limit different flow maps.

We observe that the family of solutions for which $T(c) = 0$ is singled out by the property that $\mathrm{Fl}(t,\cdot)_\sharp \mathscr{L}^1 \ll \mathscr{L}^1$ for a.e. $t \in (0,\infty)$, so no concentration of trajectories occurs at the origin. Indeed we integrate in time the identity

$$0 = \mathrm{Fl}(t,\cdot)\mathscr{L}^1(\{0\}) = \mathscr{L}^1(\{x_0 : \mathrm{Fl}(t,x_0) = 0\})$$

and use the Fubini's theorem to obtain

$$0 = \int \mathscr{L}^1(\{t : \mathrm{Fl}(t,x_0) = 0\})\, \mathrm{d}x_0.$$

This means that for $\mathscr{L}^1$-a.e. x_0 the solution $\mathrm{Fl}(\cdot,x_0)$ does not stay at 0 for a strictly positive set of times.

If we make the absolute continuity condition quantitative, we are led to the notion of Regular Lagrangian Flow.

Definition 0.0.1 (Regular Lagrangian flow)**.** *We say that* $\mathrm{Fl}\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ *is a Regular Lagrangian flow if:*

i) Fl *is a Borel map;*

ii) for $\mathscr{L}^n$*-a.e.* $x \in \mathbb{R}^n$ *the function* $t \mapsto \mathrm{Fl}(t,x)$ *is an absolutely continuous solution to the Cauchy problem* (0.5)*;*

iii) there exists a constant $C \in [0,\infty)$ *satisfying* $\mathrm{Fl}(t,\cdot)_\sharp\mathscr{L}^n \le C\mathscr{L}^n$ *for all* $t \in [0,T]$.

Arguing as before, we have that also in this case for any Lebesgue negligible Borel set $N \subset (0,T)\times\mathbb{R}^n$ the set $\{t \in (0,T) : (t,\mathrm{Fl}(t,x))\}$ is $\mathscr{L}^1$-negligible for $\mathscr{L}^n$-a.e. $x \in \mathbb{R}^n$. Hence this second condition is global, since we are selecting a family of solutions for the ODE in such a way that for all times the trajectories do not concentrate 'too much' with respect to $\mathscr{L}^n$.

In particular the property in (*iii*) implies that condition (*ii*) is actually invariant under modifications of the vector field X in $\mathscr{L}^{n+1}$-negligible sets.

Therefore one of the main results in Ambrosio-Di Perna-Lions theory is the following theorem, that ensures existence and uniqueness of the Regular Lagrangian flow:

Theorem 0.0.2. *Let* $T \in (0,\infty)$ *and* $(X_t)_{t\in[0,T]}$ *be a family of time dependent vector fields such that*

$$X \in L^1((0,T); W^{1,1}_{\mathsf{loc}}(\mathbb{R}^n;\mathbb{R}^n)), \qquad \mathrm{div}X \in L^1((0,T); L^\infty(\mathbb{R}^n)),$$

and that

$$\frac{|X|}{1+|x|} \in L^1((0,T); L^1(\mathbb{R}^n)) + L^1((0,T); L^\infty(\mathbb{R}^n)).$$

Then there exists a unique Regular Lagrangian flow Fl^X *associated to* X*, namely if* Fl^X *and* $\overline{\mathrm{Fl}}^X$ *are two Regular Lagrangian flows, then* $\mathrm{Fl}^X(\cdot,x) = \overline{\mathrm{Fl}}^X(\cdot,x)$ *for* $\mathscr{L}^n$*-a.e.* $x \in \mathbb{R}^n$.

In [15] Ambrosio and Trevisan extend the theory of well-posedness for the continuity equation and the theory of flows to the setting of metric measure spaces $(\mathrm{M},\mathsf{d},\mathfrak{m})$, where they prove that if $\{X_t\}_{t\in(0,T)}$ is a time-dependent family of Sobolev vector fields (with some more hypotheses on their divergence and covariant derivative), then there exists a unique flow associated to X_t. Here with the term 'flow' we refer to a family of absolutely continuous maps $\{\mathrm{Fl}(\cdot,x)\}_{x\in\mathrm{M}}$ from $[0,T]$ to M such that

(i) it is a Borel map;

(i) $\mathrm{Fl}(\cdot, x)$ solves the possibly non-autonomous ODE associated to X_t for $\mathfrak{m}$-a.e. $x \in \mathrm{M}$;

(ii) the measures $\mathrm{Fl}(t, \cdot)_\sharp \mathfrak{m}$ are absolutely continuous with respect to $\mathfrak{m}$ and have uniformly bounded densities.

Actually in this non smooth setting, where no local coordinates are available, first of all we have to give a meaning to the notion of 'vector field', as well as a proper definition of solution to the ODE.

In paper [15] the problem is studied from an Eulerian point of view, which is interesting for understanding the well-posedness of continuity equations, and from the Lagrangian one, that leads to the notion of solution to the ODE and allows to relate the well-posedness of the continuity equation to the existence and uniqueness of the flow (in the same spirit of [2] and [4]).

A generalization to RCD spaces of a Bochner result. In the third chapter of this thesis we prove a generalization to the non-smooth setting of RCD spaces of a classical result in Riemannian geometry. This theorem, due to Bochner, links an hypothesis on the first cohomology group of a smooth Riemannian manifold with non-negative Ricci curvature to a rigidity result on the manifold, in the following way:

Theorem 0.0.3 (Bochner)**.** *Let (M, g) be a compact and complete Riemannian manifold with* $\mathrm{Ric} \geq 0$ *and dimension equal to N. Let $\mathcal{H}^1_{dR}(M)$ be the first group in Hodge cohomology. Then:*

1. $\mathsf{dim}\mathcal{H}^1_{dR}(M) \leq N$*;*
2. *if* $\mathsf{dim}\mathcal{H}^1_{dR}(M) = N$*, then (M, g) is a flat torus.*

First of all we observe that the construction of differential calculus on RCD spaces explained by Gigli in [36] (and introduced above) provides a suitable extension of all the notions proper of Riemannian geometry to this new setting. Indeed the vocabulary proposed there allows to speak of vector fields, k-forms, covariant derivative, Hodge Laplacian and cohomology groups $\mathcal{H}^k_{\mathrm{dR}}$. In particular, in the same paper, a version of Hodge theorem has been proved in this non-smooth setting, so that we know that cohomology classes are in correspondence with their unique harmonic representative.

Instead, a notion that cannot be directly generalized to the setting of RCD spaces is the one of dimension. In this direction we first need a preliminary result, proved in [36], which says that given a generic L^2-normed L^∞-module $\mathscr{M}$ over a space $(\mathrm{M}, \mathsf{d}, \mathfrak{m})$, there exists a unique, up to negligible sets, Borel partition $(E_i)_{i \in \mathbb{N} \cup \{\infty\}}$ of M such that for every $i \in \mathbb{N}$ the restriction of $\mathscr{M}$ to E_i has dimension i, and for no $F \subset E_\infty$ with positive measure the restriction of $\mathscr{M}$ to F has finite dimension. Then we consider such partition (E_i) for $\mathscr{M}$ being the tangent module of the space M and think to the dimension of tangent module on E_i as the dimension of the space M on the same set. Hence we shall call $\mathsf{dim}_{\min}(\mathrm{M})$ (resp. $\mathsf{dim}_{\max}(\mathrm{M})$) the minimal (resp. supremum) of indexes $i \in \mathbb{N} \cup \{\infty\}$ such that $\mathfrak{m}(E_i) > 0$. With this notation, in [36] the following result has been obtained:

Theorem 0.0.4. *Let* $(\mathrm{M}, \mathsf{d}, \mathfrak{m})$ *be a* $\mathsf{RCD}^*(0, \infty)$ *space. Then* $\dim(\mathcal{H}^k_{dR})(\mathrm{M}) \leq \mathsf{dim}_{\min}(\mathrm{M})$*. Moreover, if* $\mathsf{dim}(\mathcal{H}^k_{dR})(\mathrm{M}) > 0$ *then* $\mathfrak{m}(\mathrm{M}) < \infty$*.*

We have now all the tools to generalize the proof of (i) in Theorem 0.0.3 to the framework of RCD spaces, since it closely follows the one in the Riemannian world. As a matter of fact, the proof for the first part of Theorem 0.0.3 is based on the fact that under these assumptions every harmonic 1-form must be parallel and so determined by its value at any given point $x \in M$. Since Hodge theorem ensures that the k-the cohomology group is isomorphic to the space of harmonic k-forms, the claim follows. Hence, the proof of Theorem 0.0.4 mimics this argument: every harmonic form is proved to be parallel, so that the dimension of the first cohomology group is bounded by the number of independent (co)vector fields we can find on any region of our space. Notice that thanks to this result we can remove also the compactness assumption from the hypothesis: indeed, in the terminology of [36], harmonic forms are by definition in L^2; this explains why if we can find any non-zero harmonic form, then the measure of the whole space must be finite.

However, without any additional assumption, it is possible that $\mathsf{dim}_{\min}(\mathrm{M}) = \infty$ and in this case the result above is empty. Actually a natural condition on the metric measure space $(\mathrm{M}, \mathsf{d}, \mathfrak{m})$ that ensures that the dimension $\mathsf{dim}_{\min}(\mathrm{M})$ is finite is not only that it is a $\mathsf{RCD}^*(K, \infty)$ space, but rather a $\mathsf{RCD}^*(K, N)$ one: indeed in this case $\mathsf{dim}_{\min}(\mathrm{M}) \leq \mathsf{dim}_{\max}(\mathrm{M}) \leq N$ (we refer to [45] for a proof of the second inequality). This means that Theorem 0.0.4 can be restated as:

Proposition 0.0.5. *Let* $(\mathrm{M}, \mathsf{d}, \mathfrak{m})$ *be a* $\mathsf{RCD}^*(0, N)$ *space. Then* $\dim(\mathcal{H}^1_{dR})(\mathrm{M}) \leq N$.

Therefore we have found a perfect analogue of point (i) in Theorem 0.0.3 in this new setting.

We turn then to the proof of (ii) in Theorem 0.0.3: the typical argument in the Riemannian setting starts with the observation that if an N-dimensional manifold admits N independent parallel vector fields, then such manifold must be flat. Hence its universal cover, equipped with the pullback of the metric tensor, must be the Euclidean space and the fundamental group $\pi_1(M)$ has to act on it via isometries. Since $M \sim \mathbb{R}^N/\pi_1(M)$, the conclusion follows just showing that $\pi_1(M) \sim \mathbb{Z}^N$, which can be obtained by considerations about the structure of the isometries of $\mathbb{R}^N$ and the fact that $\mathbb{R}^N/\pi_1(M)$ is, by assumption, compact and smooth (a complete proof can be found for example in [58]).

Unlike the proof of Theorem 0.0.4, here we cannot easily adapt the 'smooth arguments' to the setting of metric measure spaces having non negative Ricci curvature and finite dimension: the problem is that it is not known whether RCD spaces admit a universal cover or not (there are some recent results in this direction in [55], but it is not clear whether they can really be used for our current purposes).

In order to complete the analogy between the smooth and non-smooth setting we have then to prove the following:

Theorem 0.0.6. *Let* $(\mathrm{M}, \mathsf{d}, \mathfrak{m})$ *be a* $\mathsf{RCD}^*(0, N)$ *space such that* $\dim(\mathcal{H}^1_{dR})(\mathrm{M}) = N$ *(so that in particular* N *is integer). Then it is isomorphic to a flat torus.*

Here 'isomorphic' means that there exists a measure preserving isometry between our given space and the torus equipped with its Riemannian distance and a constant multiple of the induced volume measure.

Actually we point out that this result is new even for smooth manifolds, because the compactness of the space is not assumed there.

Briefly, the strategy we pursue in the proof of Theorem 0.0.6 is based on the following steps:

- We start studying the flow of an harmonic vector field on our space and, using the fact that in particular such vector field must be parallel and divergence-free, we prove that in accordance with the smooth case such flow is made of measure preserving isometries. Here by 'flow' we intend in fact 'Regular Lagrangian Flow' in the sense of Ambrosio-Trevisan, as described above. We remark that our appears to be the first application of Ambrosio-Trevisan theory to vector fields which are not gradients.

- We prove then that given two such vector fields X and Y, for their flows Fl_t^X and Fl_s^Y we have the formula $\mathrm{Fl}_t^X \circ \mathrm{Fl}_s^Y = \mathrm{Fl}_1^{tX+sY}$ for any $t, s \in \mathbb{R}$.
- Our assumption on the cohomology group of the space $(\mathrm{M}, \mathsf{d}, \mathfrak{m})$ grants that there are exactly N independent and orthogonal vector fields $X_1, \ldots, X_N$ which are parallel and divergence-free, hence we can define the map $\mathsf{T}\colon \mathrm{M} \times \mathbb{R}^N \to \mathrm{M}$ by

$$(x, a_1, \ldots, a_N) \quad \mapsto \quad \mathrm{Fl}_{a_1}^{X_1}(\cdots \mathrm{Fl}_{a_N}^{X_N}(x)).$$

  What previously proved ensures that this map can be seen as an action of $\mathbb{R}^N$ on M by isomorphisms.

The analysis of the properties of the map T leads to the desired isomorphism with the torus. In particular we prove that its action is transitive: to obtain this we have sharpened the calculus tools available in the non-smooth setting and, in particular, we have analyzed the structure of the (co)tangent modules on product spaces.

A direct proof of the strong maximum principle in $\mathsf{RCD}^*(K,N)$ spaces, $N \in [1,\infty)$. In the context of analysis in metric measure spaces it is by now well understood that a doubling condition and a Poincaré inequality are sufficient to derive the basics of elliptic regularity theory. In particular, one can obtain the Harnack inequality for harmonic functions which in turn implies the strong maximum principle. We refer to [22] for an overview on the topic and detailed bibliography.

Since $\mathsf{RCD}^*(K,N)$ spaces are, for finite N, doubling (cf. [68]) and supporting a Poincaré inequality (cf. [60]), the above applies (see [38] and [42] for the details). Still, given that in fact such spaces are much more regular than general doubling&Poincaré ones, one might wonder whether there is a simpler proof of the strong maximum principle.

In Chapter 4 we show that this is actually the case: out of the several arguments available in the Euclidean space, the one based on the estimates for the Laplacian of the squared distance carries over to such non-smooth context rather easily. We emphasize that such argument is, with only minor variations, the original one of Hopf appeared in [47] (the so called 'boundary point lemma' about the sign of external derivative at a maximum point at the boundary, also due to Hopf, appeared later in [48]).

In order to carry on the proof, we need to know that given a closed set C, for 'many' points $x \notin C$ there is a unique $y \in C$ minimizing the distance from x. In the Euclidean setting this is easy to prove, thanks to the strict convexity of balls, but in general metric spaces the same property can fail, even in presence of a (non-Riemannian) curvature-dimension condition. In our situation this can be proved using the existence of optimal transport maps, as proved in [44] (and actually the same result with a very similar proof has been obtained also in the recent paper [32], see Theorem 4.7 there).

A quite different direct proof of the strong maximum principle in the setting of $\mathsf{RCD}^*(K,N)$ spaces is given by Zhang and Zhu in their work [77], which is devoted to generalising in the setting of metric measure spaces the Li-Yau's local gradient estimate for solutions of the heat equation. The result in this paper is close to the Omori-Yau maximum principle ([57], [75]) and it has also some similarity with the approximate versions of the maximum principle developed in the theory of second order viscosity solutions (see for instance [49]).

CHAPTER 1

Differential calculus on RCD spaces

1.1 Overview of the chapter

In this chapter we recall the basic construction presented in [36], and we refer to this work, as well as to [33], for all the missing proofs.

We start with Section 1.2, where we introduce the basic results in the theory of L^2-normed modules, the interest being justified from the fact that the space of L^2 (co)vector fields on a Riemannian/Finslerian manifold is actually a L^2-normed module. This notion gives the possibility to speak about L^2 sections of a vector bundle, without really having the bundle, and helps the build of a differential structure on metric measure spaces, allowing to concentrate on the definition of L^2 (co)vector fields rather than a pointwise definition. In particular we point out the properties and the constructions involving Hilbert modules, that we are going to use in the construction of the second order differential calculus on $\mathsf{RCD}(K,\infty)$ spaces.

Let us consider a metric measure space $(\mathrm{M},\mathsf{d},\mathfrak{m})$ which is complete, separable and equipped with a non-negative Radon measure.

In Section 1.3 we introduce the definition of (local) Sobolev class $\mathcal{S}^2(\mathrm{M})$ and with this the definition of minimal weak gradient for functions in it. Thus using these notions we pass to the characterization of the cotangent module $L^2(T^*\mathrm{M})$, which turns out to be a L^2-normed module. For functions in $\mathcal{S}^2(\mathrm{M})$ is well defined their differential, which is an element of $L^2(T^*\mathrm{M})$ and from the properties of Sobolev functions it can be proved that it is a closed operator.

Once we have the notion of cotangent module, the tangent one $L^2(T\mathrm{M})$ can be introduced by duality (Section 1.3.3) and its elements will called vector fields. It can be easily proved that $L^2(T\mathrm{M})$ carries a canonical structure of $L^2(\mathfrak{m})$-normed module as well, so that for any vector field X the pointwise norm $|X|$ is a well defined function in $L^2(\mathfrak{m})$.

Starting from this structures, a general first order differential theory can be developed on arbitrary metric measure spaces. In particular in Section 1.3.4.1 we introduce the notion of pullback module through a map of local bounded compression. In particular, this notion allows to extend in the non-smooth setting the fact that every smooth curve γ in a Riemannian manifold has a well defined tangent vector γ_t' for every t, and its norm coincides with the metric speed of the curve. In the context of metric measure spaces this makes sense, since for any given test plan $\boldsymbol{\pi}$ it can be proved that for a.e. $t\in[0,1]$ and $\boldsymbol{\pi}$-a.e. curve γ the tangent vector γ_t' is well defined, and its norm again coincides with the metric speed $|\dot\gamma_t|$ of the curve γ at time t.

In Section 1.3.4.2 we see that (co)vector fields are transformed via maps of locally bounded deformation between metric measure spaces as in the smooth setting: we can speak of pullback of forms and these maps have a differential acting on vector fields.

Moreover we underline that the gradient of a Sobolev function is in general not uniquely defined and even if so it might not linearly depend on the function, as it happens on smooth Finsler manifolds. In Section 1.3.5 we show that spaces where the gradient $\nabla f \in L^2(T\mathrm{M})$ of a Sobolev function $f \in \mathcal{S}^2(\mathrm{M})$ is unique and linearly depends on f are exactly those which, from the Sobolev calculus point of view, resemble Riemannian manifolds among the more general Finlser ones, and can be characterized as those for which the Cheeger energy $\mathsf{E}\colon L^2(\mathfrak{m}) \to [0,+\infty]$ defined as

$$\mathsf{E}(f) := \begin{cases} \dfrac{1}{2}\int |\mathrm{D}f|^2 \,\mathrm{d}\mathfrak{m}, & \text{if } f \in \mathcal{S}^2(\mathrm{M}), \\ +\infty, & \text{otherwise} \end{cases}$$

is a Dirichlet form. These spaces are called infinitesimally Hilbertian and on them the (co)tangent module is, when seen as Banach space, an Hilbert space while its norm satisfies a pointwise parallelogram identity. Thus, by polarization, it induces a pointwise scalar product

$$L^2(T\mathrm{M}) \ni X,\ Y \quad \mapsto \quad \langle X, Y\rangle \in L^1(\mathfrak{m}),$$

which we might think if as the 'metric tensor' on our space.

Hence, with the basis provided by the general first order theory, in Section 1.4 we pass to the study of the second order differential structure arising on $\mathsf{RCD}(K,\infty)$ spaces (which are in particular infinitesimally Hilbertian). First of all we consider the following three formulas valid in a smooth Riemmanian manifold:

$$\begin{aligned} 2\mathrm{Hess}(f)(\nabla g_1, \nabla_2) &= \langle \nabla\langle \nabla f, \nabla g_1\rangle, \nabla g_2\rangle + \langle \nabla\langle \nabla f, \nabla g_1\rangle, \nabla g_2\rangle - \langle \nabla f, \nabla\langle \nabla g_1, \nabla g_2\rangle\rangle, \\ \langle \nabla_{\nabla g_2} X, \nabla g_1\rangle &= \langle \nabla\langle X, \nabla g_1\rangle, \nabla g_2\rangle - \mathrm{Hess}(g_2)(\nabla g_1, X), \\ \mathrm{d}\omega(X_1, X_2) &= X_1(\omega(X_2)) - X_2(\omega(X_1)) - \omega(\nabla_X Y - \nabla_Y X). \end{aligned}$$

The first completely characterizes the Hessian of the function f in terms of the scalar product of gradients, the second the Levi-Civita connection in terms of the Hessian and the scalar product of gradients, while the third the exterior differentiation of a 1-form via previously defined objects. Therefore their generalization in the setting of RCD spaces leads to a definition of the Hessian as well as the one of the covariant/exterior derivative, which in turn allows to speak of Sobolev vector fields and Sobolev differential forms.

At this point, in Section 1.4.7, we propose a reasonable definition of de Rham cohomology groups, which comes directly from the fact that the exterior differential d is a closed operator and from the identity $\mathrm{d}^2 = 0$. It is worth to point out that in this setting the space of Sobolev forms is not known to be compactly embedded in the one of L^2 forms. Finally, in Section 1.4.8, we see how all this machinery allows to define the Ricci curvature starting from the identity in Bochner's formula. In particular we see that for a vector field X sufficiently regular we have that this notion gives the expected bound

$$\mathrm{Ric}(X,X) \geq K|X|^2\mathfrak{m},$$

on a $\mathsf{RCD}(K,\infty)$ space.

1.2 L^2-normed modules

1.2.1 Basic definitions and properties

Definition 1.2.1 ($L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-modules)**.** *A $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module, or simply a $L^2(\mathfrak{m})$-normed module, is a structure $(\mathscr{M}, \|\cdot\|, \cdot, |\cdot|)$ where* *i)* $(\mathscr{M}, \|\cdot\|)$ *is a Banach space;*

ii) $\cdot$ *is a bilinear map from* $L^\infty(\mathfrak{m}) \times \mathscr{M}$ *to* $\mathscr{M}$*, called multiplication by* $L^\infty(\mathfrak{m})$ *functions, such that*

$$f \cdot (g \cdot v) = (fg) \cdot v, \tag{1.2.1.1a}$$

$$\mathbf{1} \cdot v = v, \tag{1.2.1.1b}$$

for every $v \in \mathscr{M}$ *and* $f, g \in L^\infty(\mathfrak{m})$*, where* $\mathbf{1}$ *is the function identically equal to 1;*

iii) $|\cdot|$ *is a map from* $\mathscr{M}$ *to* $L^2(\mathfrak{m})$*, called pointwise norm, such that*

$$|v| \geq 0, \quad \mathfrak{m}\text{-}a.e. \tag{1.2.1.2a}$$

$$|fv| = |f||v|, \quad \mathfrak{m}\text{-}a.e. \tag{1.2.1.2b}$$

$$\|v\| = \sqrt{\int |v|^2 \, \mathrm{d}\mathfrak{m}}. \tag{1.2.1.2c}$$

In the following we write fv in place of $f \cdot v$ for the product with $L^\infty(\mathfrak{m})$ functions.

We observe that the Banach space $L^2(\mathfrak{m})$ itself has a natural structure of $L^\infty(\mathfrak{m})$-module, where the multiplication with a function in $L^\infty(\mathfrak{m})$ is just the pointwise one. A more interesting example is the following one, which motivates the definition of $L^2(\mathfrak{m})$-normed module.

Example 1.2.2 (L^2 vector fields as $L^2(\mathfrak{m})$-normed modules)**.** *Let* M *be a smooth Riemannian manifold equipped with a reference measure* $\mathfrak{m}$ *and with a normed vector bundle. Then the space of* $L^2(\mathfrak{m})$*-sections of the vector bundle has a natural structure of* $L^2(\mathfrak{m})$*-normed module, where the multiplication with an* $L^\infty(\mathfrak{m})$ *function is again the pointwise one.*

Directly from (1.2.1.2b) and (1.2.1.2c) it follows that

$$\|fv\| \leq \|f\|_{L^\infty} \|v\|, \tag{1.2.1.3}$$

while the pointwise norm satisfies:

$$|\lambda v| = |\lambda||v| \tag{1.2.1.4a}$$

$$|v + w| \leq |v| + |w|, \tag{1.2.1.4b}$$

$\mathfrak{m}$-a.e. for every $v, w \in \mathscr{M}$ and $\lambda \in \mathbb{R}$. As for the proof of (1.2.1.4a) we first observe that (1.2.1.1b) ensures that there is no distinction between λv intended as coming from the vector space and as the product of v with the function constantly equal to λ; then (1.2.1.4a) follows directly from (1.2.1.2b). So we turn to the proof of (1.2.1.4b). In order to do it we argue by contradiction: suppose that (1.2.1.4b) does not hold. In this case there exist $v, w \in \mathscr{M}$, $E \subset \mathrm{M}$ Borel set with $\mathfrak{m}(E) > 0$ we would have

$$|v + w| \geq c, \quad |v| \leq a, \quad |w| \leq b, \mathfrak{m}\text{-a.e. in } E$$

for some $a, b, c \in \mathbb{R}^+$ with $a + b < c$. But this is in contradiction with (1.2.1.2c) and with the fact that $\|\cdot\|$ is a norm since:

$$\begin{aligned} \|\chi_E v\| + \|\chi_E w\| = \|\chi_E |v|\|_{L^2(\mathfrak{m})} + \|\chi_E |w|\|_{L^2(\mathfrak{m})} &\leq \sqrt{\mathfrak{m}(E)}(a + b) \\ &< \sqrt{\mathfrak{m}(E)} c \leq \|\chi_E |v + w|\|_{L^2(\mathfrak{m})} \|\chi_E (v + w)\|. \end{aligned}$$

In the following for given $v, w \in \mathscr{M}$ and $E \subset \mathrm{M}$ Borel set, we shall say that $v = w$ $\mathfrak{m}$-a.e. on E if $\chi_E(v - w) = 0$ or, equivalently, if $|v - w| = 0$ $\mathfrak{m}$-a.e. on E.

Definition 1.2.3 (Isomorphism between $L^2(\mathfrak{m})$-normed modules)**.** *An isomorphisms between two $L^2(\mathfrak{m})$-normed modules is a linear bijection which preserves the norm, the product with $L^\infty(\mathfrak{m})$ functions and the pointwise norm.*

We introduce now the notion of L^0-normed module. Indeed in applications it might be necessary to deal with objects with less integrability or to have at disposal elements v of some bigger space, having a pointwise norm $|v|$, and the possibility to say that $v \in \mathscr{M}$, $\mathscr{M}$ being a L^2-normed module if and only if $|v| \in L^2(\mathfrak{m})$.

Definition 1.2.4 (L^0-normed module)**.** *A L^0-normed module is a structure $(\mathscr{M}, \tau, \cdot, |\cdot|)$ where*

i) $\cdot$ is a bilinear map from $L^0(\mathfrak{m}) \times \mathscr{M}$ to $\mathscr{M}$, called multiplication with $L^0(\mathfrak{m})$ functions, for which (1.2.1.4a) *and* (1.2.1.4b) *hold for any $f \in L^0(\mathfrak{m}), v \in \mathscr{M}$;*

ii) $|\cdot|\colon \mathscr{M} \to L^0(\mathfrak{m})$, called pointwise norm, satisfies (1.2.1.2a) *and* (1.2.1.2b) *for any $f \in L^0(\mathfrak{m}), v \in \mathscr{M}$;*

iii) for some Borel partition (E_i) of M *into sets of finite $\mathfrak{m}$-measure, $\mathscr{M}$ is complete with respect to the distance*

$$\mathsf{d}_0(v, w) := \sum_i \frac{1}{2^i \mathfrak{m}(E_i)} \int_{E_i} \min\{1, |v - w|\} \, \mathrm{d}\mathfrak{m} \tag{1.2.1.5}$$

and τ is the topology induced by the distance.

An isomorphisms of $L^0(\mathfrak{m})$-normed modules is a linear homeomorphism preserving the pointwise norm and the multiplication with L^0-functions.

It can be proved that the choice of the partition (E_i) in (*iii*) affects neither the completeness of $\mathscr{M}$, nor the topology τ.

Theorem/Definition 1.2.5 (L^0 completion of a $L^2(\mathfrak{m})$-normed module)**.** *Let $\mathscr{M}$ be a $L^2(\mathfrak{m})$-normed module. Then there exists a unique couple $(\mathscr{M}^0, \iota)$, where $\mathscr{M}^0$ is a L^0-normed module and $\iota\colon \mathscr{M} \to \mathscr{M}^0$ is linear, preserving the pointwise norm and with dense image.*

Here for unique we mean up to a unique isomorphism, namely if $(\tilde{\mathscr{M}}^0, \tilde{\iota})$ has the same properties, then there exists a unique isomorphism $\Phi\colon \mathscr{M}^0 \to \tilde{\mathscr{M}}^0$ such that $\tilde{\iota} = \Phi \circ \iota$.

proof As for the existence define $\mathscr{M}^0$ to be the metric completion of $\mathscr{M}$ with respect to the distance in (1.2.1.5) and let ι be the natural embedding. Then we get the conclusion just observing that the $L^2(\mathfrak{m})$-normed module structure of $\mathscr{M}$ can be extended by continuity and induce an L^0-normed module structure on $\mathscr{M}^0$. Uniqueness follows by construction. □

1.2.2 Dual module

Definition 1.2.6 (Dual of a module)**.** *Let $\mathscr{M}$ be a $L^2(\mathfrak{m})$-normed module. Its dual $\mathscr{M}^*$ is the space of linear continuous maps $L\colon \mathscr{M} \to L^1(\mathfrak{m})$ such that*

$$L(fv) = fL(v) \qquad \forall f \in L^\infty(\mathfrak{m}),\ v \in \mathscr{M}.$$

We endow $\mathscr{M}^*$ with the operator norm, i.e., $\|L\|_* := \sup_{v:\|v\|\le 1} \|L(v)\|_{L^1(\mathfrak{m})}$. The multiplication of $f \in L^\infty(\mathfrak{m})$ and $L \in \mathscr{M}^*$ is defined as

$$(fL)(v) := L(fv), \qquad \forall v \in \mathscr{M}.$$

Finally the pointwise norm $|L|_*$ of $L \in \mathscr{M}^*$ is defined as

$$|L|_* := \operatorname*{ess-sup}_{v:|v|\le 1\ \mathfrak{m}\text{-}a.e.} |L(v)|.$$

The only thing to check in order to prove that the structure just introduced is actually a $L^2(\mathfrak{m})$-normed module is the identity in (1.2.1.2c) (which in particular means that $|L|_*$ is in $L^2(\mathfrak{m})$), namely we want to prove that $\|L\|_* = \||L|_*\|_{L^2(\mathfrak{m})}$. Directly from the definition we see that

$$|L(v)| \le |L|_*|v| \quad \mathfrak{m}\text{-a.e.} \qquad \forall\, v \in \mathscr{M},\, L \in \mathscr{M}^*,$$

and, just integrating this inequality, we get $\|L(v)\|_{L^1(\mathfrak{m})} \le \|v\|\||L|_*\|_{L^2(\mathfrak{m})}$ that in turn shows that $\|L\|_* \le \||L|_*\|_{L^2(\mathfrak{m})}$.

We have then to prove the opposite inequality. From the basic properties of the essential supremum there exists a sequence $(v_n) \subset \mathscr{M}$ such that $|v_n| \le 1$ $\mathfrak{m}$-a.e. for every $n \in \mathbb{N}$ and with the property that $|L|_* = \sup_n |L(v_n)|$. Thus we define a new sequence $(\tilde{v}_n)$ by posing $\tilde{v}_0 := v_0$ and for any $n > 0$ we set (recursively) $A_n := \{|L(v_n)| \ge |L(\tilde{v}_{n-1})|\}$ and $\tilde{v}_n := \chi_{A_n} v_n + \chi_{A_n^C}\tilde{v}_{n-1}$: this new sequence is such that $|\tilde{v}_n| \le 1$ $\mathfrak{m}$-a.e., while the sequence $(|L(\tilde{v}_n)|)$ is increasing and converges $\mathfrak{m}$-a.e. to $|L|_*$. Now we notice that for any function $f \in L^2 \cap L^\infty(\mathfrak{m})$ we have $\|f\tilde{v}_n\| = \||f\tilde{v}_n|\|_{L^2(\mathfrak{m})} \le \|f\|_{L^2(\mathfrak{m})}$ and so

$$\int |f||L(\tilde{v}_n)|\,\mathrm{d}\mathfrak{m} = \int |L(f\tilde{v}_n)|\,\mathrm{d}\mathfrak{m} \le \|f\tilde{v}_n\|\|L\|_* = \|f\|_{L^2(\mathfrak{m})}\|L\|_* \qquad \forall\, n \in \mathbb{N}.$$

The monotone convergence theorem ensures that the integral on the left hand side goes to $\int |f||L|_*\,\mathrm{d}\mathfrak{m}$ as $n \to \infty$, thus we can pass to the limit, obtaining

$$\int |f||L|_*\,\mathrm{d}\mathfrak{m} \le \|f\|_{L^2(\mathfrak{m})}\|L\|_*.$$

Since this is true for every $f \in L^2 \cap L^\infty(\mathfrak{m})$, we conclude that $\||L|_*\|_{L^2(\mathfrak{m})} \le \|L\|_*$ as desired.

At this point we consider $\mathscr{M}'$, the dual of $\mathscr{M}$ seen as Banach space: $\mathscr{M}'$ is then the Banach space of linear and continuous maps from $\mathscr{M}$ to $\mathbb{R}$ equipped with its canonical norm $\|\cdot\|_{\prime}$. Integration grants the existence of a map $\mathrm{Int}\colon \mathscr{M}^* \to \mathscr{M}'$ which associates to every $L \in \mathscr{M}^*$ the operator $\mathrm{Int}(L) \in \mathscr{M}'$ defined by

$$\mathrm{Int}(L)(v) := \int L(v)\,\mathrm{d}\mathfrak{m}, \quad \forall\, v \in \mathscr{M}.$$

We have that

Proposition 1.2.7. *The map* Int *is a bijective isometry, namely for every $L \in \mathscr{M}^*$ it holds $\|L\|_* = \|\mathrm{Int}(L)\|_{\prime}$.*

Hence the Hahn-Banach theorem ensures that for any $v \in \mathscr{M}$ there exists an element $\ell \in \mathscr{M}'$ with $\|\ell\|_{\prime} = \|v\|$ and $|\ell(v)| = \|v\|^2$. Now if we take $L = \mathrm{Int}^{-1}(v)$ we find

$$\|v\|^2 = \ell(v) = \int L(v)\,\mathrm{d}\mathfrak{m} \le \int |L|_*|v|\,\mathrm{d}\mathfrak{m} \le \||v|\|_{L^2(\mathfrak{m})}\||L|_*\|_{L^2(\mathfrak{m})} = \|v\|\|L\|_* = \|v\|\|\ell\|_{\prime} = \|v\|^2$$

which means that the inequality above is actually an equality and so

$$|L|_* = |v| \quad \text{and} \quad L(v) = |v|^2 \quad \mathfrak{m}\text{-a.e.} \tag{1.2.2.1}$$

Therefore the natural L^∞-linear embedding $\mathscr{J} : \mathscr{M} \to \mathscr{M}^{**}$, which sends v to the map $L \mapsto L(v)$, preserves the pointwise norm. We can see it just observing that for every v and L it holds $|\mathscr{J}(v)(L)| = |L(v)| \le |v||L|_*$ and so $|\mathscr{J}(v)|_{**} \le |v|$, while the opposite inequality can be achieved by considering L such that (1.2.2.1) is satisfied.

Definition 1.2.8 (Reflexive Module). *A $L^2(\mathfrak{m})$-normed module is called reflexive if the map $\mathscr{J}$ is surjective.*

We recall the following useful criterion which allows to recognize elements in the dual module via their action on a generating space:

Proposition 1.2.9. *Let $\mathscr{M}$ be a $L^2(\mathfrak{m})$-normed module, $V \subset \mathscr{M}$ a vector subspace which generates $\mathscr{M}$ and $L\colon V \to L^1(\mathfrak{m})$ a linear map. Suppose that for some $g \in L^2(\mathfrak{m})$ it holds*

$$|L(v)| \leq g|v| \quad \mathfrak{m}\text{-a.e.} \quad \forall\, v \in V. \tag{1.2.2.2}$$

Then there exists a unique $\tilde{L} \in \mathscr{M}^*$ such that $\tilde{L}(v) = L(v)$ for every $v \in V$ and for such $\tilde{L}$ we have $|\tilde{L}|_* \leq g$ $\mathfrak{m}$-a.e..

It is useful to have a version of this result also for $L^0(\mathfrak{m})$-modules. First of all we introduce the dual of an $L^0(\mathfrak{m})$-module $\mathscr{M}_0$ by setting

$$\begin{aligned}\mathscr{M}_0^* := \{\mathsf{T}\colon \mathscr{M}_0 \to L^0(\mathfrak{m}) :&\mathsf{T} \text{ is linear and there exists } \ell \in L^0(\mathfrak{m})\\ &\text{such that } |\mathsf{T}(v)| \leq \ell|v| \quad \mathfrak{m}\text{-a.e., } \forall v \in \mathscr{M}_0\}.\end{aligned}$$

Proposition 1.2.10. *Let $\mathscr{M}_0$ be a $L^0(\mathfrak{m})$-module and let $V \subset \mathscr{M}_0$ be a vector subspace which generates $\mathscr{M}_0$, namely the $L^0(\mathfrak{m})$-linear combinations are dense in $\mathscr{M}_0$. If $\mathsf{T}\colon V \to L^1(\mathfrak{m})$ is a linear map such that for some $g \in L^0(\mathfrak{m})$ it holds*

$$|\mathsf{T}(v)| \leq g|v| \quad \mathfrak{m}\text{-a.e.} \quad \forall\, v \in V, \tag{1.2.2.3}$$

then there exists a unique $\tilde{\mathsf{T}} \in \mathscr{M}_0^*$ such that it holds $\tilde{\mathsf{T}}(v) = \mathsf{T}(v)$ for every $v \in V$ and it still satisfies the bound in (1.2.2.3) for every $v \in \mathscr{M}_0$.

proof We start observing that if we want the extension $\tilde{\mathsf{T}}$ of T to be $L^0(\mathfrak{m})$-linear the only possible definition of $\tilde{\mathsf{T}}$ for $v = \sum_i \chi_{A_i} v_i$, where (A_i) is a finite Borel partition of M and $(v_i) \subset V$, is the following:

$$\tilde{\mathsf{T}}\left(\sum_i \chi_{A_i} v_i\right) := \sum_i \chi_{A_i} \mathsf{T}(v_i). \tag{1.2.2.4}$$

Hence for $\tilde{\mathsf{T}}$ defined in this way the bound (1.2.2.3) provides

$$|\tilde{\mathsf{T}}(v)| = \sum_i \chi_{A_i} |\mathsf{T}(v_i)| \leq \sum_i \chi_{A_i} g|v_i| = g\left|\sum_i \chi_{A_i} v_i\right| = g|v|.$$

This in particular shows that the definition (1.2.2.4) is actually well-posed, in the sense that $\tilde{\mathsf{T}}(v)$ depends only on v and not on the way to represent it as $\sum_i \chi_{A_i} v_i$. Moreover this $\tilde{\mathsf{T}}$ is continuous and so the fact that the the set of v's of the form $\sum_i \chi_{A_i} v_i$ is dense in $\mathscr{M}_0$ ensures that there exists a unique extension of $\tilde{\mathsf{T}}$ to a continuous operator $\tilde{\mathsf{T}}\colon \mathscr{M}_0 \to L^0(\mathfrak{m})$, which is $L^0(\mathfrak{m})$-linear and still satisfies the bound in (1.2.2.3) for every $v \in \mathscr{M}_0$. □

Finally we conclude this section with the following proposition which ensures that the operations of taking the dual and of taking the L^0-completion commutes:

Proposition 1.2.11. *Let $\mathscr{M}$ be a $L^2(\mathfrak{m})$-normed module. Then the dual pairing $\mathscr{M} \times \mathscr{M}^* \to L^1(\mathfrak{m})$ uniquely extends to a continuous duality pairing $\mathscr{M}^0 \times (\mathscr{M}^*)^0 \to L^0(\mathfrak{m})$. Furthermore, if $L\colon \mathscr{M}^0 \to L^0(\mathfrak{m})$ is such that for some $g \in L^0(\mathfrak{m})$ it holds*

$$|L(v)| \leq g|v| \quad \mathfrak{m}\text{-a.e.} \quad \forall\, v \in \mathscr{M}^0, \tag{1.2.2.5}$$

then $L \in (\mathscr{M}^*)^0$ (in the sense of the just introduced pairing).

1.2.3 Local dimension

In this section we introduce the notion of dimension for a $L^2(\mathfrak{m})$-normed module. We start with few definitions:

Definition 1.2.12 (Local independence). *Let $\mathscr{M}$ be a $L^2(\mathfrak{m})$-normed module and $A \in \mathscr{B}(\mathrm{M})$ be such that $\mathfrak{m}(A) > 0$. We say that a finite family $v_1, \dots, v_n \in \mathscr{M}$ is independent on A provided the identity*

$$\sum_{i=1}^{n} f_i v_i = 0, \qquad \mathfrak{m}\text{-a.e. on } A$$

holds only if $f_i = 0$ $\mathfrak{m}$-a.e. on A for every $i = 1, \dots, n$.

Definition 1.2.13. *Let $\mathscr{M}$ be a $L^2(\mathfrak{m})$-normed module, $V \subset \mathscr{M}$ a subset and $A \in \mathscr{B}(\mathrm{M})$.*

The span of V on A, denoted by $Span_A(V)$, is the subset of $\mathscr{M}$ made of vectors v concentrated on A satisfying the following property: there exists a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathscr{B}(\mathrm{M})$ of Borel disjoint sets such that $A = \cup_i A_i$ and for every n there are m_n elements $v_{1,n}, \dots, v_{m_n,n} \in \mathscr{M}$ and m_n functions $f_{1,n}, \dots, f_{m_n,n} \in L^\infty(\mathfrak{m})$ such that

$$\chi_{A_n} v = \sum_{i=1}^{m_n} f_{i,n} v_{i,n}.$$

We refer to $Span_A(V)$ as the space spanned by V on A, while we refer to the closure $\overline{Span_A(V)}$ of $Span_A(V)$ as the space generated by V on A.

We say that $\mathscr{M}$ is *finitely generated* if there exists a finite family $v_1, \dots, v_n$ spanning $\mathscr{M}$ on the whole M and *locally finite generated* if there is a partition (E_i) of M such that $\mathscr{M}|_{E_i}$ is finitely generated for every $i \in \mathbb{N}$.

We point out that these definitions are all invariant by isomorphisms: indeed given $\mathscr{M}_1, \mathscr{M}_2$, two modules on M, $T\colon \mathscr{M}_1 \to \mathscr{M}_2$ an isomorphism of modules and $A \subset V$, then if $v_1, \dots, v_n \in \mathscr{M}_1$ are independent on A, also $T(v_1), \dots, T(v_n) \in \mathscr{M}_2$ are independent on A.

Definition 1.2.14 (Local basis and dimension). *We say that a finite family $v_1, \dots, v_n$ is a basis on $A \in \mathscr{B}(\mathrm{M})$ provided it is independent on A and $Span_A(\{v_1, \dots, v_n\}) = \mathscr{M}|_A$.*

If $\mathscr{M}$ admits a basis of cardinality n on A, we say that it has dimension n on A or that the local dimension of $\mathscr{M}$ on A is n. If $\mathscr{M}$ has not dimension n for each $n \in \mathbb{N}$, we say that it has infinite dimension.

Using the standard arguments of linear algebra we can prove that the definition of dimension is well defined, meaning that if $v_1, \dots, v_n$ generates $\mathscr{M}$ on A and $w_1, \dots, w_m$ are independent on A, then $n \geq m$. In particular, if both $v_1, \dots, v_n$ and $w_1, \dots, w_m$ are basis of $\mathscr{M}$ on A, we have that $n = m$.

Therefore we can introduce the following useful result:

Proposition 1.2.15 (Dimensional decomposition). *Let $\mathscr{M}$ be a L^2-normed module. Then there exists a unique partition $\{E_i\}_{i\in\mathbb{N}\cup\{\infty\}}$ of M such that the following holds:*

i) for every $i \in \mathbb{N}$ such that $\mathfrak{m}(E_i) > 0$, $\mathscr{M}$ has dimension i on E_i;

ii) for every $E \subset E_\infty$ with $\mathfrak{m}(E) > 0$, $\mathscr{M}$ has infinite dimension on E.

It can be proved that for a L^2-normed module the space $\mathrm{Span}_A(\{v_1, \dots, v_n\})$ is actually closed. This fact together with the Proposition 1.2.15 give us a powerful characterization of the structure of $\mathscr{M}$ as well as the following reflexivity result:

Theorem 1.2.16. *Let $\mathscr{M}$ be a $L^2(\mathfrak{m})$-normed module, $A \in \mathscr{B}(\mathrm{M})$ and suppose that the local dimension of $\mathscr{M}$ on A is n. Then the local dimension of the dual module $\mathscr{M}^*$ on A is also n. In particular this means that if $\mathscr{M}$ is locally finitely generated, then it is reflexive.*

1.2.4 Hilbert modules

Definition 1.2.17 (Hilbert modules). *We say that an $L^2(\mathfrak{m})$-module $\mathscr{M}$ is an Hilbert module provided that $\mathscr{M}$ seen as Banach space is an Hilbert space.*

On a given Hilbert module $\mathscr{H}$ we define the *pointwise scalar product* $\mathscr{H} \times \mathscr{H} \ni (v, w) \mapsto \langle v, w\rangle \in L^1(\mathfrak{m})$ as

$$\langle v, w\rangle := \frac{1}{2}\left(|v+w|^2 - |v|^2 - |w|^2\right),$$

and the standard polarization argument ensures that such map satisfies the properties

$$\begin{aligned} \langle f_1 v_1 + f_2 v_2, w\rangle &= f_1\langle v_1, w\rangle + f_2\langle v_2, w\rangle, \\ |\langle v, w\rangle| &\le |v||w|, \\ \langle v, w\rangle &= \langle w, v\rangle \\ \langle v, v\rangle &= |v|^2 \end{aligned} \tag{1.2.4.1}$$

$\mathfrak{m}$-a.e., for every $v_1, v_2, v, w \in \mathscr{H}$ and $f_1, f_2 \in L^\infty(\mathfrak{m})$.

Hence we have the following result:

Proposition 1.2.18 (Riesz theorem for Hilbert modules and reflexivity). *Let $\mathscr{H}$ be an Hilbert module and consider the map sending $v \in \mathscr{H}$ to $L_v \in \mathscr{H}^*$ given by $L_v(w) := \langle v, w\rangle$.*

Then this map is an isomorphism of modules. In particular this means that Hilbert modules are reflexive.

The results in Section 1.2.3 provide a quite complete characterization of structure of Hilbert modules. As a matter of fact to a given Hilbert space we can associate the Hilbert module $L^2(\mathrm{M}, H)$ of L^2 maps from M to H, i.e., of maps $v\colon \mathrm{M} \to H$ such that $\|v\|^2_{L^2(\mathrm{M},H)} := \int |v|^2(x)\mathrm{d}\mathfrak{m}(x) < \infty$. It is clear that $L^2(\mathrm{M}, H)$ is a $L^2(\mathfrak{m})$-normed module, the multiplication with a function in $L^\infty(\mathfrak{m})$ being simply the pointwise one, and actually the module $(L^2(\mathrm{M}, H), \|\cdot\|_{L^2(\mathrm{M},H)})$ is Hilbert.

Thus we fix an infinite dimensional separable Hilbert space H together with a sequence of subspaces $V_i \subset H$, $i \in \mathbb{N}$, such that $\mathrm{dim} V_i = i$ for every $i \in \mathbb{N}$. Now to each partition $\{E_i\}_{i\in\mathbb{N}\cup\{\infty\}}$ of M we associate the Hilbert module $\mathscr{H}(\{E_i\}; H, \{V_i\})$ made of elements $v \in L^2(\mathrm{M}, H)$ such that

$$v(x) \in V_i, \quad \mathfrak{m}\text{-a.e. on } E_i, \quad \forall\, i \in \mathbb{N}.$$

Since $\mathscr{H}(\{E_i\}; H, \{V_i\})$ is a submodule of $L^2(\mathrm{M}, H)$, it is an Hilbert module itself.

Therefore we have the following structural result:

Theorem 1.2.19 (Structural characterization of separable Hilbert modules). *Let $\mathscr{H}$ be a separable Hilbert module. Then there exists a unique partition $\{E_i\}_{i\in\mathbb{N}\cup\{\infty\}}$ of M such that $\mathscr{H}$ is isomorphic to $\mathscr{H}(\{E_i\}; H, \{V_i\})$.*

Remark 1.2.20. It is important to observe that Theorem 1.2.19 does not say that every separable Hilbert module is of the form $\mathscr{H}(\{E_i\}; H, \{V_i\})$. Actually this statement is in general wrong: if we consider the Hilbert module of L^2 vector fields on a Riemannian manifold M, this is isometric to $L^2(\mathrm{M}, \mathbb{R}^{\mathsf{dim}\mathrm{M}})$ but the choice of the isomorphism corresponds to a Borel choice of an orthogonal basis in a.e. point of M, which is possible but in general not intrinsic.

1.2.5 Tensor product of Hilbert modules

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two Hilbert modules on M and denote by $\mathscr{H}_1 \otimes_{\mathrm{Alg}} \mathscr{H}_2$ their tensor product as L^∞-modules: $\mathscr{H}_1 \otimes_{\mathrm{Alg}} \mathscr{H}_2$ can be seen as the space of formal finite sums of objects of the form $v_1 \otimes v_2$ with $(v_1, v_2) \mapsto v_1 \otimes v_2$ being L^∞-bilinear.

Now, using $\langle \cdot, \cdot \rangle_i$ the pointwise scalar product on $\mathscr{H}_i$, $i = 1, 2$, we introduce the L^∞-bilinear and symmetric map : from $[\mathscr{H}_1 \otimes_{\mathrm{Alg}} \mathscr{H}_2]^2$ to $L^0(\mathfrak{m})$ defined by

$$(v_1 \otimes v_2) : (v_1' \otimes v_2') := \langle v_1, v_1' \rangle_1 \langle v_2, v_2' \rangle_2$$

and extending it by L^∞-bilinearity. This definition is well-posed and the resulting map is positively definite, that is, for any $A \in \mathscr{H}_1 \otimes_{\mathrm{Alg}} \mathscr{H}_2$ and for any Borel set $E \subset \mathrm{M}$ it holds

$$\begin{aligned} &A : A \geq 0 \quad \mathfrak{m}\text{-a.e.} \\ &A : A = 0 \quad \mathfrak{m}\text{-a.e. on } E \quad \iff \quad A = 0 \ \mathfrak{m}\text{-a.e. on } E. \end{aligned}$$

The *Hilbert-Schmidt* pointwise norm is given by

$$|A|_{\mathrm{HS}} := \sqrt{A : A} \in L^0(\mathfrak{m})$$

while the tensor product norm by

$$\|A\|_{\mathscr{H}_1 \otimes \mathscr{H}_2} := \sqrt{\int |A|^2_{\mathrm{HS}} \, \mathrm{d}\mathfrak{m}} \ \in \ [0, +\infty].$$

We can now introduce the following:

Definition 1.2.21 (Tensor product of Hilbert modules)**.** *The space $\mathscr{H}_1 \otimes \mathscr{H}_2$ is defined as the completion of*

$$\Big\{ A \in \mathscr{H}_1 \otimes_{Alg} \mathscr{H}_2 : \ \|A\|_{\mathscr{H}_1 \otimes \mathscr{H}_2} < \infty \Big\}$$

with respect to the tensor product norm $\| \cdot \|_{\mathscr{H}_1 \otimes \mathscr{H}_2}$.

The multiplication by L^∞ functions in $\mathscr{H}_1 \otimes_{\mathrm{Alg}} \mathscr{H}_2$ induces by continuity a multiplication by L^∞ functions on $\mathscr{H}_1 \otimes \mathscr{H}_2$ which together with the definition of pointwise norm $| \cdot |_{\mathrm{HS}}$ shows that $\mathscr{H}_1 \otimes \mathscr{H}_2$ comes with the structure of L^2-normed module. Furthermore $\mathscr{H}_1 \otimes \mathscr{H}_2$ is still a Hilbert module, since $| \cdot |_{\mathrm{HS}}$ satisfies the pointwise parallelogram identity and by a truncation argument we can prove that in the case in which both $\mathscr{H}_1$ and $\mathscr{H}_2$ are separable, then so is $\mathscr{H}_1 \otimes \mathscr{H}_2$.

Let us focus on the case in which $\mathscr{H}_1 = \mathscr{H}_2 = \mathscr{H}$. In this hypothesis the tensor product will be denoted $\mathscr{H}^{\otimes 2}$ and the map $v_1 \otimes v_2 \mapsto v_2 \otimes v_1$ on $\mathscr{H}_1 \otimes_{\mathrm{Alg}} \mathscr{H}_2$ induces an automorphism $A \mapsto A^t$ on $\mathscr{H}^{\otimes 2}$, called transposition. Then for every $A \in \mathscr{H}^{\otimes 2}$ we can define the symmetric and antisymmetric part of A by posing

$$A_{\mathrm{sym}} := \frac{A + A^t}{2} \quad \text{and} \quad A_{\mathrm{Asym}} := \frac{A - A^t}{2}.$$

In particular it holds

$$|A|^2_{\mathrm{HS}} = |A_{\mathrm{Sym}}|^2_{\mathrm{HS}} + |A_{\mathrm{Asym}}|^2_{\mathrm{HS}} \quad \mathfrak{m}\text{-a.e.}, \forall \ A \in \mathscr{H}^{\otimes 2}.$$

1.2.6 Exterior power of a Hilbert module

Let $\mathscr{H}$ be an Hilbert module and let $k \in \mathbb{N}$. If $k = 0$ we set $\Lambda^0 \mathscr{H} := L^2(\mathfrak{m})$, while if $k > 0$ we consider $\mathscr{H}^{\otimes k} := \underbrace{\mathscr{H} \otimes \cdots \otimes \mathscr{H}}_{k\text{-times}}$ the tensor product of k-copies of $\mathscr{H}$. The k-th exterior power $\Lambda^k \mathscr{H}$ of $\mathscr{H}$ is defined as the quotient of $\mathscr{H}^{\otimes k}$ with respect to the space of L^∞-linear combinations of elements of the form $v_1 \otimes \cdots \otimes v_k$ with $v_i = v_j$ for at least two different indexes i, j. We denote by $v_1 \wedge \cdots \wedge v_k$ the image of $v_1 \otimes \cdots \otimes v_k$ under the quotient map and we endow $\Lambda^k \mathscr{H}$ with the quotient pointwise scalar product given by (up to a multiplication by the factor $k!$)

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle := \det(\langle v_i, w_j \rangle), \quad \mathfrak{m}\text{-a.e.}$$

Such scalar product is positively definite, meaning that for every $\omega \in \Lambda^k \mathscr{H}$

$$\begin{aligned} &\langle \omega, \omega \rangle \geq 0 \quad \mathfrak{m}\text{-a.e.} \\ &\langle \omega, \omega \rangle = 0 \quad \mathfrak{m}\text{-a.e. on } E \quad \iff \quad \chi_E \omega = 0. \end{aligned}$$

Hence it makes sense to define the pointwise norm $|\omega| := \sqrt{\langle \omega, \omega \rangle} \in L^2(\mathfrak{m})$.

With the same arguments used for the case of tensor product we see that $\Lambda^k \mathscr{H}$ is an Hilbert module and that if $\mathscr{H}$ is separable, then so is $\Lambda^k \mathscr{H}$ for any $k \in \mathbb{N}$.

We conclude observing that the map

$$(v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_{k'}) \quad \mapsto \quad v_1 \wedge \cdots \wedge v_k \wedge w_1 \wedge \cdots \wedge w_{k'}$$

is a bilinear and continuous map, called wedge product, from $\Lambda^k \mathscr{H} \times \Lambda^{k'} \mathscr{H}$ to $\Lambda^{k+k'} \mathscr{H}$.

1.3 First order differential structure of general metric measure spaces

1.3.1 Sobolev functions on metric measure spaces

Let $(\mathrm{M}, \mathsf{d}, \mathfrak{m})$ be a metric measure space such that (M, d) is complete and separable and $\mathfrak{m}$ is a non-negative Radon measure. By $C([0,1], \mathrm{M})$ we denote the space of continuous curves with value in M endowed with the sup-norm. Note that since (M, d) is complete and separable, $C([0,1], \mathrm{M})$ is complete and separable as well. For $t \in [0,1]$ we consider the evaluation map $e_t \colon C([0,1], \mathrm{M}) \to \mathrm{M}$ defined by

$$e_t(\gamma) := \gamma_t, \qquad \forall \gamma \in C([0,1], \mathrm{M}).$$

A curve $\gamma \colon [0,1] \to \mathrm{M}$ is said to be absolutely continuous provided there exists a function $f \in L^1(0,1)$ such that

$$\mathsf{d}(\gamma_s, \gamma_t) \leq \int_t^s f(r)\, \mathrm{d}r, \qquad \forall t, s \in [0,1],\ t < s. \tag{1.3.1.1}$$

The metric speed $t \mapsto |\dot{\gamma}_t| \in L^1(0,1)$ of an absolutely continuous curve γ is defined as the essential-infimum of all the functions $f \in L^1(0,1)$ for which (1.3.1.1) holds.

It is useful to introduce the corresponding notions for locally integrable objects. By $L^2_{\mathsf{loc}}(\mathrm{M})$ we mean the space of (equivalence classes with respect to $\mathfrak{m}$-a.e. equality of) Borel functions $f : \mathrm{M} \to \mathbb{R}$ such that $\chi_B f \in L^2(\mathrm{M})$ for every bounded Borel set $B \subset \mathrm{M}$. A curve $t \mapsto f_t \in L^2_{\mathsf{loc}}(\mathrm{M})$

will be called continuous (resp. absolutely continuous, Lipschitz, C^1) provided for any bounded Borel set $B \subset \mathrm{M}$ the curve $t \mapsto \chi_B f_t \in L^2(\mathrm{M})$ is continuous (resp. absolutely continuous, Lipschitz, C^1).

There are several equivalent ways to define Sobolev functions on a metric measure space, we follow an approach proposed in [8]. We start with the following definition:

Definition 1.3.1 (Test Plans). *Let $\boldsymbol{\pi} \in \mathscr{P}(C([0,1],\mathrm{M}))$. We say that $\boldsymbol{\pi}$ is a test plan if there exists a constant $C(\boldsymbol{\pi})$ such that*

$$(\mathrm{e}_t)_*\boldsymbol{\pi} \le C\mathfrak{m} \qquad \forall t \in [0,1],$$
$$\iint_0^1 |\dot\gamma_t|^2 \,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi} < \infty.$$

Notice that any test plan must be concentrated on absolutely continuous curves, since if γ is not absolutely continuous it holds $\int_0^1 |\dot\gamma_t|^2 \mathrm{d}t = +\infty$.

Definition 1.3.2 (The Sobolev class). *The Sobolev class $\mathcal{S}^2(\mathrm{M})$ (resp. $\mathcal{S}^2_{\mathsf{loc}}(\mathrm{M})$) is the space of all the functions $f \in L^0(\mathfrak{m})$ for which there exists a non-negative $G \in L^2(\mathfrak{m})$ (resp. $G \in L^2_{\mathsf{loc}}(\mathfrak{m})$), called weak upper gradient, such that*

$$\int |f(\gamma_1) - f(\gamma_0)|\,\mathrm{d}\boldsymbol{\pi}(\gamma) \le \iint_0^1 G(\gamma_t)|\dot\gamma_t|\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma) \qquad \textit{for every test plan } \boldsymbol{\pi}. \tag{1.3.1.2}$$

We remark that since $(e_0)_\sharp\boldsymbol{\pi}, (e_1)_\sharp\boldsymbol{\pi} \ll \mathfrak{m}$ the integral in the left hand side of (1.3.1.2) is well defined, meaning that it depends only on the equivalence class of the function f and not on its chosen representative. Similarly for the right hand side.

It turns out that $f \in \mathcal{S}^2_{\mathsf{loc}}(\mathrm{M})$ and G is a weak upper gradient if and only if for every test plan $\boldsymbol{\pi}$ we have that for $\boldsymbol{\pi}$-a.e. γ the map $t \mapsto f(\gamma_t)$ is in $W^{1,1}(0,1)$ and

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma_t)\right| \le G(\gamma_t)|\dot\gamma_t| \qquad a.e.\ t \in [0,1]. \tag{1.3.1.3}$$

From this characterization it follows that there exists a minimal weak upper gradient in the $\mathfrak{m}$-a.e. sense: it will be called minimal weak upper gradient and denoted by $|\mathrm{D}f|$.

Moreover with a simple cut-off argument, (1.3.1.3) shows that $f \in \mathcal{S}^2_{\mathsf{loc}}(\mathrm{M})$ if and only if $\eta f \in \mathcal{S}^2(\mathrm{M})$ for every η Lipschitz and with bounded support.

We briefly recall the basic properties of Sobolev functions and minimal weak upper gradients that we are going to use in the following:

- *Local semicontinuity of minimal weak gradients*: Let $(f_n) \subset \mathcal{S}^2(\mathrm{M})$ and $f \in L^0(\mathfrak{m})$ be such that $f_n \to f$ $\mathfrak{m}$-a.e. as $n \to \infty$. Suppose that $(|\mathrm{D}f_n|)$ converges to some $G \in L^2(\mathfrak{m})$ weakly in $L^2(\mathfrak{m})$. Then $f \in \mathcal{S}^2(\mathrm{M})$ and $|\mathrm{D}f| \le G$ $\mathfrak{m}$-a.e.
- *Vector space structure*: $\mathcal{S}^2(\mathrm{M})$ is a vector space and

$$|\mathrm{D}(\alpha f + \beta g)| \le |\alpha||\mathrm{D}f| + |\beta||\mathrm{D}g|, \qquad \text{for any, } f, g \in \mathcal{S}^2(\mathrm{M}),\ \alpha, \beta \in \mathbb{R}.$$

- *Algebra structure*: $\mathcal{S}^2 \cap L^\infty(\mathrm{M})$ is an algebra and

$$|\mathrm{D}(fg)| \le |f||\mathrm{D}g| + |g||\mathrm{D}f|, \qquad \text{for any, } f, g \in \mathcal{S}^2 \cap L^\infty(\mathrm{M}).$$

- *Locality*: The minimal weak upper gradient is local, namely:

$$|\mathrm{D}f| = |\mathrm{D}g|, \quad \mathfrak{m}\text{-a.e. on } \{f = g\}, \quad \forall\, f, g \in \mathcal{S}^2(\mathrm{M}). \tag{1.3.1.4}$$

- *Chain rule*: For every $f \in \mathcal{S}^2(\mathrm{M})$ we have
$$|\mathrm{D}f| = 0, \quad \text{on } f^{-1}(\mathcal{N}), \quad \forall \mathcal{N} \subset \mathbb{R}, \text{ Borel with } \mathscr{L}^1(\mathcal{N}) = 0, \tag{1.3.1.5}$$
moreover for $f \in \mathcal{S}^2(\mathrm{M})$, I an open subset of $\mathbb{R}$ such that $\mathfrak{m}(f^{-1}(\mathbb{R} \setminus I)) = 0$ and $\varphi \colon I \to \mathbb{R}$ Lipschitz, we have $\varphi \circ f \in \mathcal{S}^2(\mathrm{M})$ and
$$|\mathrm{D}(\varphi \circ f)| = |\varphi'| \circ f \; |\mathrm{D}f|, \tag{1.3.1.6}$$
where $|\varphi'| \circ f$ is defined arbitrarily at points where φ is not differentiable: indeed identity (1.3.1.5) guarantees that on $f^{-1}(\mathcal{N})$ both $|\mathrm{D}(\varphi \circ f)|$ and $|\mathrm{D}f|$ are 0 $\mathfrak{m}$-a.e., where $\mathcal{N}$ is the negligible set of points of non-differentiability of φ.

Therefore we define the Sobolev space $W^{1,2}(\mathrm{M})$ (resp. $W^{1,2}_{\mathsf{loc}}(\mathrm{M})$) by posing $W^{1,2}(\mathrm{M}) := L^2 \cap \mathcal{S}^2(\mathrm{M})$ (resp. $L^2_{\mathsf{loc}} \cap \mathcal{S}^2_{\mathsf{loc}}(\mathrm{M})$) and we endow it with the norm
$$\|f\|^2_{W^{1,2}(\mathrm{M})} := \|f\|^2_{L^2(\mathfrak{m})} + \||\mathrm{D}f|\|^2_{L^2(\mathfrak{m})}.$$
It is worth to remark that $W^{1,2}(\mathrm{M})$ is always a Banach space, but in general not an Hilbert one.

Finally, the fact that for $f \in \mathcal{S}^2_{\mathsf{loc}}(\mathrm{M})$ the truncated function $(f \wedge (-c)) \vee c$ also belongs to $\mathcal{S}^2_{\mathsf{loc}}(\mathrm{M})$ for every $c > 0$ together with a cut-off argument provide the following useful approximation result:
$$\begin{gathered}\forall f \in \mathcal{S}^2_{\mathsf{loc}}(\mathrm{M}) \text{ there exists } (f_n) \subset \mathcal{S}^2(\mathrm{M}) \text{ such that } \mathfrak{m}(\mathrm{M} \setminus \cup_n \{f = f_n\}) = 0 \\ \text{and for every } n \in \mathbb{N} \text{ the function } f_n \text{ is bounded and with bounded support.}\end{gathered} \tag{1.3.1.7}$$

1.3.2 Cotangent module and differential

Starting from the notion of minimal weak upper gradient it is possible to extract the one of differential via the following result:

Theorem/Definition 1.3.3 (Cotangent Module)**.** *There exists a unique couple $(L^0(T^*\mathrm{M}), \mathrm{d})$ with $L^0(T^*\mathrm{M})$ being a $L^0(\mathrm{M})$-normed module and $\mathrm{d} : \mathcal{S}^2_{\mathsf{loc}}(\mathrm{M}) \to L^0(T^*\mathrm{M})$ linear and such that*

(i) $|\mathrm{d}f| = |\mathrm{D}f|$ $\mathfrak{m}$-a.e. for every $f \in \mathcal{S}^2_{\mathsf{loc}}(\mathrm{M})$,

(ii) $L^0(T^*\mathrm{M})$ is generated by $\{\mathrm{d}f : f \in \mathcal{S}^2_{\mathsf{loc}}(\mathrm{M})\}$, i.e. L^0-linear combinations of objects of the form $\mathrm{d}f$ are dense in $L^0(T^*\mathrm{M})$.

Uniqueness is intended up to unique isomorphism, i.e. if $(\mathscr{M}, \mathrm{d}')$ is another such couple, then there is a unique isomorphism $\Phi \colon L^0(T^*\mathrm{M}) \to \mathscr{M}$ such that $\Phi(\mathrm{d}f) = \mathrm{d}'f$ for every $f \in \mathcal{S}^2_{\mathsf{loc}}(\mathrm{M})$.

The approximation result in (1.3.1.7) allows to replace $\mathcal{S}^2_{\mathsf{loc}}(\mathrm{M})$ with either $\mathcal{S}^2(\mathrm{M})$ or $W^{1,2}(\mathrm{M})$. In particular if we consider either $\mathcal{S}^2(\mathrm{M})$ or $W^{1,2}(\mathrm{M})$ in place of $\mathcal{S}^2_{\mathsf{loc}}(\mathrm{M})$, then it is also possible to replace the L^0-normed module with a L^2-normed module in the statement and in this case in (*ii*) 'L^0-linear' should be replaced by 'L^∞-linear'. Although the choice of the module also affects the topology considered, whence the possibility of having two different uniqueness results, the proof rests unaltered (we refer to [36] and [33] for all the details). Moreover, directly from the definitions it follows that if $(L^2(T^*\mathrm{M}), \underline{\mathrm{d}})$ is the L^2-normed modulus found in Theorem 1.3.3 using the spaces $\mathcal{S}^2(\mathrm{M})$, $W^{1,2}(\mathrm{M})$, then its L^0-completion can be fully identified with the couple $(L^0(T^*\mathrm{M}), \mathrm{d})$ given by Theorem 1.3.3, namely there is a unique linear map

$\iota : L^2(T^*\mathrm{M}) \to L^0(T^*\mathrm{M})$ sending $\underline{\mathrm{d}}f$ to $\mathrm{d}f$ and preserving the pointwise norm; in particular this map has dense image. For this reason we can avoid to use a distinguished notation for the differential coming from the L^2 formulation of the statement. We will call $L^2(T^*\mathrm{M})$ the *cotangent module* of the metric measure space $(\mathrm{M}, \mathsf{d}, \mathfrak{m})$, d the differential and we will refer to the elements of $L^2(T^*\mathrm{M})$ as *1-forms.*

We define $L^2_{\mathsf{loc}}(T^*\mathrm{M})$ as the space of 1-forms ω's such that $|\omega| \in L^2_{\mathsf{loc}}(\mathfrak{m})$.

Remark 1.3.4. Notice that (1.3.1.7) ensures also that if D is a dense subset of $W^{1,2}(\mathrm{M})$, then $\{\mathrm{d}f : f \in D\}$ generates $L^2(T^*\mathrm{M})$. Hence if $W^{1,2}(\mathrm{M})$ is separable, so is $L^2(T^*\mathrm{M})$.

All the expected properties of the differential keep holding also in this non-smooth setting. Indeed the following holds:

- *Locality*: For every $f, g \in \mathcal{S}^2(\mathrm{M})$ we have
$$\mathrm{d}f = \mathrm{d}g \quad \mathfrak{m}\text{-a.e. on } \{f = g\}. \tag{1.3.2.1}$$
- *Chain rule*: For every $f \in \mathcal{S}^2(\mathrm{M})$ and $\varphi \in \mathrm{Lip} \cap C^1(\mathbb{R})$ we have $\varphi \circ f \in \mathcal{S}^2(\mathrm{M})$ and
$$\mathrm{d}(\varphi \circ f) = \varphi' \circ f \ \mathrm{d}f \tag{1.3.2.2}$$
- *Leibniz rule*: For every $f, g \in L^\infty \cap \mathcal{S}^2(\mathrm{M})$ we have $fg \in \mathcal{S}^2(\mathrm{M})$
$$\mathrm{d}(fg) = f \ \mathrm{d}g + g \ \mathrm{d}f. \tag{1.3.2.3}$$

Actually the following useful result guarantees that:

Theorem 1.3.5. *Let $\mathscr{M}$ a $L^\infty(\mathfrak{m})$-module and $L\colon \mathcal{S}^2(\mathrm{M}) \to \mathscr{M}$ a linear map continuous with respect to the Sobolev norm. Then L satisfies the locality property if and only if it satisfies the chain rule, if and only if it satisfies the Leibniz rule.*

Finally an important result about the differential is related to its closure:

Theorem 1.3.6 (Closure of d)**.** *Let $(f_n) \subset \mathcal{S}^2(\mathrm{M})$ be a sequence $\mathfrak{m}$-a.e. converging to some function $f \in L^0(\mathrm{M})$ and such that $(\mathrm{d}f_n)$ converges to some $\omega \in L^2(T^*\mathrm{M})$ in the weak topology of $L^2(T^*\mathrm{M})$ seen as Banach space. Then $f \in \mathcal{S}^2(\mathrm{M})$ and $\mathrm{d}f = \omega$.*

1.3.3 Tangent Module

Definition 1.3.7 (Tangent Module)**.** *The tangent module $L^2(T\mathrm{M})$ is defined as the dual of the cotangent module $L^2(T^*\mathrm{M})$ and its elements are called vector fields.*

As we have seen in Section 1.2.2, $L^2(T\mathrm{M})$ is an $L^2(\mathfrak{m})$-normed module. Despite the fact that we introduce it by duality, to keep consistence with the notation used in the smooth setting, we shall denote the pointwise norm in $L^2(T\mathrm{M})$ as $|\cdot|$, rather than $|\cdot|_*$, and the duality pairing between $\omega \in L^2(T^*\mathrm{M})$ and $X \in L^2(T\mathrm{M})$ as $\omega(X)$.

The space of vector fields $L^0(T\mathrm{M})$ is defined as the dual of the L^0-normed module $L^0(T^*\mathrm{M})$. Equivalently, it is the L^0-completion of the dual $L^2(T\mathrm{M})$ of the L^2-normed module $L^2(T^*\mathrm{M})$ (see [36], [33]). $L^2_{\mathsf{loc}}(T\mathrm{M}) \subset L^0(T\mathrm{M})$ is the space of X's such that $|X| \in L^2_{\mathsf{loc}}(\mathrm{M})$.

In particular in the case in which $L^2(T^*\mathrm{M})$ and $L^2(T\mathrm{M})$ are Hilbert modules, they are canonically isomorphic via the Riesz Theorem for Hilbert modules (see Proposition 1.2.18): this map, called *musical isomorphism*, is denoted by

$$\flat\colon L^2(T^*\mathrm{M}) \to L^2(T\mathrm{M}) \quad \text{and} \quad \sharp\colon L^2(T\mathrm{M}) \to L^2(T^*\mathrm{M}) \tag{1.3.3.1}$$

where $X^\flat(Y) := \langle X, Y\rangle$ and $\langle \omega^\sharp, X\rangle := \omega(X)$ $\mathfrak{m}$-a.e. for every $X, Y \in L^2(T\mathrm{M})$ and $\omega \in L^2(T^*\mathrm{M})$.

The maps $\flat$ and $\sharp$ uniquely extend to continuous maps from $L^0(T\mathrm{M})$ to $L^0(T^*\mathrm{M})$ and from $L^0(T^*\mathrm{M})$ to $L^0(T\mathrm{M})$, respectively.

We turn then to the definition of derivation, namely a linear operator which satisfies the Leibniz rule and which associates to every function in $\mathcal{S}^2(\mathrm{M})$ a $\mathfrak{m}$-a.e. defined function.

Definition 1.3.8 (L^2 derivations)**.** *A L^2-derivation is a linear map $L\colon \mathcal{S}^2(\mathrm{M}) \to L^1(\mathfrak{m})$ for which there exists $g \in L^2(\mathfrak{m})$ such that*

$$|L(f)| \le g|\mathrm{D}f|, \quad \forall\, f \in \mathcal{S}^2(\mathrm{M}). \tag{1.3.3.2}$$

To see that for such a defined object Leibniz and chain rule hold, we just point out that given a derivation L, $f, g \in \mathcal{S}^2(\mathrm{M})$, we have

$$|L(f-g)| \le g|\mathrm{D}(f-g)| = 0, \quad \mathfrak{m}\text{-a.e. on } \{f = g\},$$

and so

$$L(f) = L(g), \quad \mathfrak{m}\text{-a.e. on } \{f = g\}. \tag{1.3.3.3}$$

Hence since it holds

$$\|L(f)\|_{L^1(\mathfrak{m})} \le \|g\|_{L^2(\mathfrak{m})} \||\mathrm{D}f|\|_{L^2(\mathfrak{m})}, \quad \forall\, f \in \mathcal{S}^2(\mathrm{M}),$$

the locality property (1.3.3.3) and Theorem 1.3.5 applied to the module $L^1(\mathfrak{m})$ guarantee that indeed Leibniz and chain rule hold.

In particular, for any $f \in \mathcal{S}^2(\mathrm{M})$ we have $|\mathrm{D}f| \in L^2(\mathfrak{m})$ and so it is quite natural to think about L^2-derivations, simply by duality.

In the next result we are going to prove that vector fields and derivations are actually two different points of view of the same concept:

Theorem 1.3.9 (Derivations and vector fields)**.** *For any vector field $X \in L^2(T\mathrm{M})$ the map $X \circ \mathrm{d}\colon \mathcal{S}^2(\mathrm{M}) \to L^1(\mathfrak{m})$ is a derivation.*

Conversely, for any derivation L there exists a unique vector field $X \in L^2(T\mathrm{M})$ such that the diagram

$$\begin{tikzcd}
\mathcal{S}^2(\mathrm{M}) \arrow[r, "\mathrm{d}"] \arrow[dr, "L"'] & L^2(T^*\mathrm{M}) \arrow[d, "X"] \\
 & L^1(\mathfrak{m})
\end{tikzcd}$$

commutes.

proof The map $X \circ \mathrm{d}$ is linear and satisfies

$$|(X \circ \mathrm{d})(f)| = |\mathrm{d}f(X)| \le |X||\mathrm{d}f|_* = |X||\mathrm{D}f|, \quad \mathfrak{m}\text{-a.e.} \quad \forall\, f \in \mathcal{S}^2(\mathrm{M}).$$

Since $|X| \in L^2(\mathfrak{m})$, the first claim is proved.

As for the second claim, let L be a derivation and consider the linear map from $V := \{\mathrm{d}f : f \in \mathcal{S}^2(\mathrm{M})\}$ to $L^1(\mathfrak{m})$ defined by

$$\mathrm{d}f \mapsto \tilde{L}(\mathrm{d}f) := L(f).$$

Inequality (1.3.3.2) and the identity $|\mathrm{d}f|_* = |\mathrm{D}f|$ ensure that this map is well defined, i.e. $\tilde{L}(\mathrm{d}f)$ depends only on $\mathrm{d}f$ and not on f, and that

$$|\tilde{L}(\mathrm{d}f)| \leq g|\mathrm{d}f|.$$

The conclusion follows directly from Proposition 1.2.9 recalling that V generates $L^2(T^*\mathrm{M})$. □

We are now ready to introduce the definition of divergence of a vector field as the adjoint of the differential:

Definition 1.3.10 (Divergence)**.** *We say that $X \in L^2(T\mathrm{M})$ (resp. in $L^2_{\mathsf{loc}}(T\mathrm{M})$) has divergence in L^2 (resp. in L^2_{loc}), and we write $X \in \mathrm{D}(\mathrm{div})$ (resp. $X \in \mathrm{D}(\mathrm{div}_{\mathsf{loc}})$), provided there is $h \in L^2(\mathfrak{m})$ (resp. $h \in L^2_{\mathsf{loc}}(\mathfrak{m})$) such that*

$$\int fh\,\mathrm{d}\mathfrak{m} = -\int \mathrm{d}f(X)\,\mathrm{d}\mathfrak{m}, \qquad \forall\, f \in W^{1,2}(\mathrm{M}). \tag{1.3.3.4}$$

In this case we call h the divergence of X and denote it by $\mathrm{div}(X)$*.*

In particular the density of $W^{1,2}(\mathrm{M})$ in $L^2(\mathfrak{m})$ ensures that there is at most one h satisfying (1.3.3.4), which means that the divergence is unique.

The linearity of the differential implies that $\mathrm{D}(\mathrm{div})$ is a vector space and that the divergence is a linear operator.

The Leibniz rule for differentials immediately gives the Leibniz rule for the divergence, namely:

$$\begin{aligned}&\text{if } X \in \mathrm{D}(\mathrm{div}) \text{ and } f \in L^\infty \cap \mathcal{S}^2(\mathrm{M}) \ \text{ with } \ |\mathrm{d}f|_* \in L^\infty(\mathfrak{m})\\ &\text{we have } \ fX \in \mathrm{D}(\mathrm{div}) \ \text{ and } \ \mathrm{div}(fX) = \mathrm{d}f(X) + f\mathrm{div}(X).\end{aligned} \tag{1.3.3.5}$$

Indeed in these hypothesis on the function f and on the vector field X, we have $\mathrm{d}f(X) + f\mathrm{div}(X) \in L^2(\mathfrak{m})$ and for any $g \in W^{1,2}(\mathrm{M})$ the product fg is actually in $W^{1,2}(\mathrm{M})$ with

$$-\int fg\mathrm{div}(X)\,\mathrm{d}\mathfrak{m} = \int \mathrm{d}(fg)(X)\,\mathrm{d}\mathfrak{m} = \int g\mathrm{d}f(X) + \mathrm{d}g(fX)\,\mathrm{d}\mathfrak{m}, \tag{1.3.3.6}$$

which is the claim. We underline that this in particular means that if $X \in \mathrm{D}(\mathrm{div})$ and $g \in \mathrm{Lip}_b(\mathrm{M})$, then $gX \in \mathrm{D}(\mathrm{div})$ with

$$\mathrm{div}(gX) = \mathrm{d}g(X) + g\mathrm{div}(X).$$

1.3.4 Pullback of modules and forms

Here we shall extend the constructions of pullback module and pullback of 1-forms provided in [36] (see also [33]) to maps which are locally of bounded compression/deformation. For this purpose, it is technically convenient to work with L^0-normed modules rather than with L^2-normed ones.

1.3.4.1 Pullback module through a map of local bounded compression

Definition 1.3.11 (Maps of local bounded compression). *Let* $(\mathrm{M}_1, \mathfrak{m}_1)$ *and* $(\mathrm{M}_2, \mathfrak{m}_2)$ *be two* σ*-finite measured spaces. We say that* $\varphi\colon \mathrm{M}_1 \to \mathrm{M}_2$ *is a map of local bounded compression provided for every* $B \subset \mathscr{B}(\mathrm{M}_1)$ *with* $\mathfrak{m}_1(B) < +\infty$ *there exists a constant* $C_B \geq 0$ *such that*

$$\varphi_*\big(\mathfrak{m}_1{}_{|_B}\big) \leq C_B \mathfrak{m}_2. \tag{1.3.4.1}$$

Theorem/Definition 1.3.12. *Let* $(\mathrm{M}_1, \mathfrak{m}_1)$ *and* $(\mathrm{M}_2, \mathfrak{m}_2)$ *be two* σ*-finite measured spaces,* $\varphi\colon \mathrm{M}_1 \to \mathrm{M}_2$ *be of local bounded compression and* $\mathscr{M}$ *a* $L^0(\mathrm{M}_2)$*-normed module. Then there exists a unique couple* $([\varphi^*]\mathscr{M}, [\varphi^*])$ *where* $[\varphi^*]\mathscr{M}$ *is a* $L^0(\mathrm{M}_1)$*-module and* $[\varphi^*] : \mathscr{M} \to [\varphi^*]\mathscr{M}$ *is a linear map such that:*

1) $|[\varphi^*]\, v| = |v| \circ \varphi$ $\mathfrak{m}_1$*-a.e. for every* $v \in \mathscr{M}$*,*
2) $[\varphi^*]\mathscr{M}$ *is generated by* $\{[\varphi^*]\,(v) : v \in \mathscr{M}\}$*.*

Uniqueness is intended up to unique isomorphism, i.e.: if $\big(\widetilde{[\varphi^*]\mathscr{M}}, [\tilde{\varphi}^*]\big)$ *is another such couple, then there exists a unique isomorphism* $\Phi\colon [\varphi^*]\mathscr{M} \to \widetilde{[\varphi^*]\mathscr{M}}$ *such that* $[\tilde{\varphi}^*] = [\varphi^*] \circ \Phi$*.*

proof
Existence We define the 'pre-pullback' set Ppb as

$$\mathrm{Ppb} := \big\{(v_i, A_i)_{i=1,\dots,n} \,:\, n \in \mathbb{N},\ (A_i) \subset \mathscr{B}(\mathrm{M}_1) \text{ is a partition of } \mathrm{M}_1 \text{ and } v_i \in \mathscr{M}\ \forall i = 1,\dots,n\big\}$$

and an equivalence relation on it by declaring $(v_i, A_i) \sim (w_j, B_j)$ provided

$$|v_i - w_j| \circ \varphi = 0 \quad \mathfrak{m}_1\text{-a.e. on } A_i \cap B_j \qquad \forall\, i, j.$$

It s readily verified that it is actually an equivalence relation on Ppb: we shall denote by $[(v_i, A_i)]$ the equivalence class of (v_i, A_i).

We endow $\mathsf{Ppb}/\sim$ with a vector space structure by putting

$$\begin{aligned} [(v_i, A_i)] + [(w_j, B_j)] &:= [(v_i + w_j, A_i \cap B_j)] \\ \lambda[(v_i, A_i)] &:= [(\lambda v_i, A_i)] \end{aligned}$$

for every $[(v_i, A_i)], [(w_j, B_j)] \in \mathsf{Ppb}/\sim$ and $\lambda \in \mathbb{R}$. Notice that these are well defined. Moreover, we define a pointwise norm on $\mathsf{Ppb}/\sim$ and a multiplication with simple functions by putting

$$\begin{aligned} |[(v_i, A_i)]| &:= \sum_i \chi_{A_i} |v_i| \circ \varphi \\ g[(v_i, A_i)] &:= [(\alpha_j v_i, A_i \cap E_j)] \qquad \text{for} \qquad g = \sum_j \alpha_j \chi_{E_j}. \end{aligned}$$

Again, these are easily seen to be well defined; then we fix a partition $(E_i) \subset \mathscr{B}(\mathrm{M}_1)$ of M_1 made of sets of finite $\mathfrak{m}_1$-measure and define the distance d_0 on $\mathsf{Ppb}/\sim$ as

$$\mathrm{d}_0([(v_i, A_i)], [(w_j, B_j)]) := \sum_{k \in \mathbb{N}} \frac{1}{2^k \mathfrak{m}_1(E_k)} \int_{E_k} \big|[(v_i, A_i)] - [(w_j, B_j)]\big| \,\mathrm{d}\mathfrak{m}_1.$$

We then define the space $[\varphi^*]\mathscr{M}$ as the completion of $(\mathrm{Ppb}/\sim, \mathsf{d}_0)$, equipped with the induced topology and the pullback map $[\varphi^*] : \mathscr{M} \to [\varphi^*]\mathscr{M}$ as $[\varphi^*]v := [v, \mathrm{M}_1]$. The (in)equalities

$$\begin{aligned} |[(v_i + w_j, A_i \cap B_j)]| &\leq |[(v_i, A_i)]| + |[(w_j, B_j)]|, \\ |\lambda[(v_i, A_i)]| &= |\lambda||[(v_i, A_i)]|, \\ |g[(v_i, A_i)]| &= |g||[(v_i, A_i)]|, \end{aligned}$$

valid $\mathfrak{m}_1$-a.e. for every $[(v_i, A_i)], [(w_j, B_j)] \in \mathrm{Ppb}/\sim$, $\lambda \in \mathbb{R}$ and simple function g grant that the vector space structure, the pointwise norm and the multiplication by simple functions can all be extended by continuity to the whole $[\varphi^*]\mathscr{M}$ and it is then clear with these operations such space is a L^0-normed module.

Property (1) then follows by the very definitions of pullback map and pointwise norm, while property (2) from the fact that $\mathsf{Ppb}/\sim$ is dense in $[\varphi^*]\mathscr{M}$ and the typical element $[v_i, A_i]$ of $\mathsf{Ppb}/\sim$ is equal to $\sum_i \chi_{A_i}[\varphi^*]v_i$.

Uniqueness The requirement for $\Phi\colon [\varphi^*]\mathscr{M} \to \widetilde{[\varphi^*]\mathscr{M}}$ to be $L^0(\mathrm{M}_1)$-linear and such that $[\tilde{\varphi}^*] = [\varphi^*] \circ \Phi$ force the definition

$$\Phi(V) := \sum_i \chi_{A_i}[\tilde{\varphi}^*]v_i \qquad \text{for} \qquad V = \sum_i \chi_{A_i}[\varphi^*]v_i. \tag{1.3.4.2}$$

The identity

$$|\Phi(V)| = \sum_i \chi_{A_i}|[\tilde{\varphi}^*]v_i| \overset{(1)\text{ for }\widetilde{[\varphi^*]\mathscr{M}}}{=} \sum_i \chi_{A_i}|v_i| \circ \varphi \overset{(1)\text{ for }[\varphi^*]\mathscr{M}}{=} \sum_i \chi_{A_i}|[\varphi^*]v_i| = |V|$$

shows in particular that the definition of $\Phi(V)$ is well-posed, i.e. it depends only on V and not on the particular way to represent it as sum. It also shows that it preserves the pointwise norm and thus it is continuous. Since the space of V's of the form $\sum_i \chi_{A_i}[\varphi^*]v_i$ is dense in $[\varphi^*]\mathscr{M}$ (property (2) for $[\varphi^*]\mathscr{M}$), such Φ can be uniquely extended to a continuous map on the whole $[\varphi^*]\mathscr{M}$ and such extension is clearly linear, continuous, and preserves the pointwise norm. By definition, it also holds $\Phi(gV) = g\Phi(V)$ for g simple and $V = \sum_i \chi_{A_i}[\varphi^*]v_i$ and thus by approximation we see that the same holds for general $g \in L^0(\mathrm{M}_1)$ and $V \in [\varphi^*]\mathscr{M}$.

It remains to show that the image of Φ is the whole $\widetilde{[\varphi^*]\mathscr{M}}$: this follows from the fact that elements of the form $\sum_i \chi_{A_i}[\tilde{\varphi}^*]v_i$, which by definition are in the image of Φ, are dense in $\widetilde{[\varphi^*]\mathscr{M}}$ by property (2) for $\widetilde{[\varphi^*]\mathscr{M}}$. □

We shall now provide an explicit representation of such pullback module in the case when φ is a projection.

Thus let $(\mathrm{M}_1, \mathfrak{m}_1)$ and $(\mathrm{M}_2, \mathfrak{m}_2)$ be two σ-finite measured spaces and let $\mathscr{M}$ be a $L^0(\mathrm{M}_1)$-module over M_1.

We shall denote by $L^0(\mathrm{M}_2, \mathscr{M})$ the space of (equivalence classes up to $\mathfrak{m}_2$-a.e. equality of) strongly measurable (namely, Borel and essentially separably valued) functions from M_2 to $\mathscr{M}$ and claim that such space canonically carries the structure of $L^0(\mathrm{M}_1 \times \mathrm{M}_2)$-normed module. The multiplication of an element in $L^0(\mathrm{M}_2, \mathscr{M})$ by a function $f \in L^0(\mathrm{M}_1 \times \mathrm{M}_2)$ is defined as the map $\mathrm{M}_2 \ni x_2 \mapsto f(\cdot, x_2) \cdot v(\cdot, x_2) \in \mathscr{M}$. By approximating f in $L^0(\mathrm{M}_1 \times \mathrm{M}_2)$ with functions having finite range it is easy to check that $x_2 \mapsto f(\cdot, x_2) \cdot v(\cdot, x_2)$ has separable range if $x_2 \mapsto v(\cdot, x_2)$ does, so that this definition is well-posed. Similarly, the pointwise norm of $v \in L^0(\mathrm{M}_2, \mathscr{M})$ is obtained composing the map $x_2 \mapsto v(\cdot, x_2) \in \mathscr{M}$ with the pointwise norm on $\mathscr{M}$, thus providing an element of $L^0(\mathrm{M}_2, L^0(\mathrm{M}_1)) \sim L^0(\mathrm{M}_1 \times \mathrm{M}_2)$. Finally, we use such pointwise norm to define the topology of $L^0(\mathrm{M}_2, \mathscr{M})$ as in point (*iii*) of Definition 1.2.4.

In particular sequences converging in this topology are made of maps $v_n(\cdot, x_2)$ which are $\mathfrak{m}_2$-a.e. converging and that with these definitions $L^0(\mathrm{M}_2, \mathscr{M})$ is indeed a $L^0(\mathrm{M}_1 \times \mathrm{M}_2)$-normed module.

In what will come next, we shall often implicitly use the identification:

Proposition 1.3.13 ($[\pi_1]^*\mathscr{M}$ is isomorphic to $L^0(\mathrm{M}_2, \mathscr{M})$)**.** *Let* $(\mathrm{M}_1, \mathfrak{m}_1)$ *and* $(\mathrm{M}_2, \mathfrak{m}_2)$ *be two* σ*-finite measured spaces,* $\mathscr{M}$ *a* $L^0(\mathrm{M}_1)$*-module over* M_1 *and* $\pi_1 : \mathrm{M}_1 \times \mathrm{M}_2 \to \mathrm{M}_1$ *the canonical projection.*

Then there exists a unique isomorphism from $[\pi_1{}^*](\mathscr{M})$ *to* $L^0(\mathrm{M}_2,\mathscr{M})$ *which for every* $v\in\mathscr{M}$ *sends* $[\pi_1^*]v$ *to the function in* $L^0(\mathrm{M}_2,\mathscr{M})$ *constantly equal to* v.

proof For $v\in\mathscr{M}$ let $\hat v\in L^0(\mathrm{M}_2,\mathscr{M})$ be the function constantly equal to v. It is clear that $|\hat v|=|v|\circ\pi_1$ $\mathfrak{m}_1\times\mathfrak{m}_2$-a.e. Moreover, from the fact that functions in $L^0(\mathrm{M}_2,\mathscr{M})$ are essentially separably valued it follows by standard means in vector-space integration that $\{\hat v : v\in\mathscr{M}\}$ generate the whole $L^0(\mathrm{M}_2,\mathscr{M})$.

The conclusion comes from Theorem 1.3.12. □

Proposition 1.3.14 (Universal property of the pullback)**.** *Let* $\mathscr{M}_0$ *be a* $L^0(\mathfrak{m}_1)$*-module,* $\mathscr{N}_0$ *be a* $L^0(\mathfrak{m}_2)$*-module,* $\varphi\colon\mathrm{M}_2\to\mathrm{M}_1$ *be a map of locally bounded compression and* $\mathsf{T}\colon\mathscr{M}\to\mathscr{N}$ *linear and such that for some* $C>0$ *it holds*

$$|\mathsf{T}(v)|\le C|v|\circ\varphi,\qquad \mathfrak{m}_2\text{-}a.e.$$

Then there exists a unique $L^0(\mathfrak{m})$*-linear and continuous map* $\hat{\mathsf{T}}\colon\varphi^*\mathscr{M}_0\to\mathscr{N}_0$ *such that*

$$\hat{\mathsf{T}}(\varphi^*v)=\mathsf{T}(v)\qquad\forall\ v\in\mathscr{M}_0.$$

proof Let us consider the space $V:=\{\varphi^*v : v\in\mathscr{M}_0\}$ and recall that this generates the $L^0(\mathfrak{m})$-module $\varphi^*\mathscr{M}_0$ (meaning that the set of $L^0(\mathfrak{m}_1)$ linear combinations of elements in V is dense in $\varphi^*\mathscr{M}_0$). Then if we consider the map $L\colon V\to\mathscr{N}_0$ defined by $L(\varphi^*v):=\mathsf{T}(v)$ and we argue as in the proof of Proposition 1.2.10, we get the conclusion. □

Directly from this proposition we see that if $\varphi\colon\mathrm{M}_2\to\mathrm{M}_1$ and $\psi\colon\mathrm{M}_3\to\mathrm{M}_2$ are both of locally bounded compression and $\mathscr{M}_0$ is a $L^0(\mathfrak{m}_1)$ module, then $\psi^*\varphi^*\mathscr{M}_0$ can be canonically identified to $(\psi\circ\varphi)^*\mathscr{M}_0$ via the isomorphism which sends $\psi^*\varphi^*v$ to $(\psi\circ\varphi)^*v$ for every $v\in\mathscr{M}_0$.

Remark 1.3.15. In the case in which φ is invertible with inverse of locally bounded deformation, we have that φ^* is actually bijective. Moreover the right composition with φ gives an isomorphism of $L^0(\mathfrak{m}_1)$ and $L^0(\mathfrak{m}_2)$ and, under this isomorphism, the two modules $\mathscr{M}_0$ and $\varphi^*\mathscr{M}_0$ can be identify (the isomorphism being φ^*).

Finally we see the duality relation between $\varphi^*\mathscr{M}_0$ and $\varphi^*\mathscr{M}_0^*$, where $\mathscr{M}_0^*$ is as usual the dual of the $L^0(\mathfrak{m})$-module $\mathscr{M}_0$ and $\varphi^*\mathscr{M}_0^*$ is its pullback.

Proposition 1.3.16. *There exists a unique* $L^0(\mathfrak{m}_2)$*-bilinear and continuous map form* $\varphi^*\mathscr{M}_0\times\varphi^*\mathscr{M}_0^*$ *to* $L^0(\mathfrak{m}_2)$ *such that*

$$\varphi^*\omega(\varphi^*v)=\omega(v)\circ\varphi,\qquad\forall\ v\in\mathscr{M}_0,\ \omega\in\mathscr{M}_0^*.\tag{1.3.4.3}$$

and for such map it holds

$$|W(V)|\le|W|_*|V|,\qquad\forall\ V\in\varphi^*\mathscr{M}_0,\ \omega\in\varphi^*\mathscr{M}_0^*.\tag{1.3.4.4}$$

proof The proof of this result is again obtained with an argument similar to the one in the proof of Proposition 1.2.10. We start considering simple elements $W\in\varphi^*\mathscr{M}_0^*$ and $V\in\varphi^*\mathscr{M}_0$ and we observe that the requirement (1.3.4.3) together with the $L^0(\mathfrak{m})$-bilinearity force the definition

$$W(V):=\sum_{i,j}\chi_{A_i\cap B_j}\omega_i(v_j)\circ\varphi,\qquad\text{for }\ W=\sum_i\chi_{A_i}\varphi^*\omega_i\ \text{ and }\ V=\sum_j\chi_{B_j}\varphi^*v_j.\tag{1.3.4.5}$$

Moreover the bound

$$|\sum_{i,j}\chi_{A_i\cap B_j}\omega_i(v_j)\circ\varphi|\le\sum_{i,j}\chi_{A_i\cap B_j}|\omega_i|\circ\varphi|v_j|\circ\varphi=\sum_i\chi_{A_i}|\omega_i|\circ\varphi\sum_j\chi_{B_j}|v_j|\circ\varphi=|W||V|$$

shows that the above definition (1.3.4.5) is well-posed and that (1.3.4.4) holds for simple elements.

Then since (1.3.4.5) ensures that $(fW)(gV) = fgW(V)$ for f, g simple functions, the density of simple elements in the respective modules allows to conclude. $\square$

The above proposition tells us that there exists a natural embedding $\mathscr{I}$ of $\varphi^*\mathscr{M}_0^*$ into $(\varphi^*\mathscr{M}_0)^*$ which sends $W \in \varphi^*\mathscr{M}_0^*$ into the map

$$\varphi^*\mathscr{M}_0 \ni V \quad \mapsto \quad W(V) \in L^0(\mathfrak{m}).$$

This embedding $\mathscr{I}$ is actually a module morphism which preserves the pointwise norm. An interesting question is then whether it is surjective: if so, this would mean that $\varphi^*\mathscr{M}_0^*$ can be identified with the dual of $\varphi^*\mathscr{M}_0$. In general the answer is negative: indeed an equivalent formulation to this question is whether the dual of $L^0(\mathrm{M}_2, \mathscr{M}_0)$ is given by $L^0(\mathrm{M}_2, \mathscr{M}_0^*)$ and this is true if and only if $\mathscr{M}_0^*$ has the Radon-Nikodym property (which in turn is ensured for example in the case in which $\mathscr{M}_0^*$ is separable).

Theorem 1.3.17 (Identification of $\varphi^*\mathscr{M}_0^*$ and $(\varphi^*\mathscr{M}_0)^*$). *Let $(\mathrm{M}_1, \mathsf{d}_1, \mathfrak{m}_1)$ and $(\mathrm{M}_2, \mathsf{d}_2, \mathfrak{m}_2)$ be two completed and separable metric spaces equipped with non-negative Borel measures finite on bounded sets and $\varphi\colon \mathrm{M}_2 \to \mathrm{M}_1$ of locally bounded compression. Assume that $\mathscr{M}_0$ is a $L^0(\mathfrak{m})$-module such that its dual $\mathscr{M}_0^*$ is separable. Then $\mathscr{I}\colon \varphi^*\mathscr{M}_0^* \to (\varphi^*\mathscr{M}_0)^*$ is surjective.*

1.3.4.2 Localized pullback of 1-forms

Definition 1.3.18 (Maps of local bounded deformation). *Let $(\mathrm{M}_1, \mathsf{d}_1, \mathfrak{m}_1)$ and $(\mathrm{M}_2, \mathsf{d}_2, \mathfrak{m}_2)$ be two metric measure spaces. We say that a map $\varphi : \mathrm{M}_1 \to \mathrm{M}_2$ is of local bounded deformation if for every bounded set $B \subset \mathrm{M}_1$ there are constants $L(B), C(B) > 0$ such that:*

$$\begin{gathered}\varphi \text{ is } L(B)\text{-Lipschitz on } B\\ \varphi_*\big(\mathfrak{m}_1|_B\big) \le C(B)\mathfrak{m}_2.\end{gathered}$$

Recalling that the local Lipschitz constant $\operatorname{lip}\varphi : \mathrm{M}_1 \to [0, \infty]$ is defined as

$$\operatorname{lip}\varphi(x) := \varlimsup_{y \to x} \frac{\mathsf{d}_2(\varphi(x), \varphi(y))}{\mathsf{d}_1(x, y)}$$

if x is not isolated, 0 otherwise, we have the following statement:

Proposition 1.3.19. *Let $(\mathrm{M}_1, \mathsf{d}_1, \mathfrak{m}_1)$ and $(\mathrm{M}_2, \mathsf{d}_2, \mathfrak{m}_2)$ be two metric measure spaces and $\varphi : \mathrm{M}_1 \to \mathrm{M}_2$ be a map of local bounded deformation. Then for any $f \in \mathcal{S}^2_{\mathsf{loc}}(\mathrm{M}_2)$, we have $f \circ \varphi \in \mathcal{S}^2_{\mathsf{loc}}(\mathrm{M}_1)$ and*

$$|\mathrm{d}(f \circ \varphi)| \le \operatorname{lip}\varphi\, |\mathrm{d}f| \circ \varphi, \qquad \mathfrak{m}_1 - a.e.. \tag{1.3.4.6}$$

proof Fix a point $\bar{x} \in \mathrm{M}_1$, let $A_n \subset C([0,1], \mathrm{M}_1)$ be defined as

$$A_n := \big\{\gamma \in C([0,1], \mathrm{M}_1) : \ \gamma_t \in B_n(\bar{x}) \ \forall t \in [0,1]\big\}$$

and notice that $\cup_n A_n = C([0,1], \mathrm{M}_1)$. Now let $\boldsymbol{\pi}$ be a test plan on M_1 and notice that for n sufficiently large the measure $\boldsymbol{\pi}_n := \boldsymbol{\pi}(A_n)^{-1}\boldsymbol{\pi}|_{A_n}$ is well defined and a test plan. By construction we have

$$(\mathrm{e}_t)_*\varphi_*\boldsymbol{\pi}_n \le \boldsymbol{\pi}(A_n)^{-1} C(B_n(\bar{x})) C(\boldsymbol{\pi})\mathfrak{m}_2 \qquad \forall t \in [0,1],$$

where $C(\boldsymbol{\pi})$ is such that $(\mathrm{e}_t)_*\boldsymbol{\pi} \leq C(\boldsymbol{\pi})\mathfrak{m}_1$ for every $t \in [0,1]$, and taking into account the trivial bound

$$\{\text{metric speed of } t \mapsto \varphi(\gamma_t)\} \leq \mathrm{lip}\,\varphi(\gamma_t)|\dot{\gamma}_t| \qquad a.e.\ t \tag{1.3.4.7}$$

we also have

$$\iint_0^1 |\dot{\gamma}_t|^2\,\mathrm{d}t\,\mathrm{d}\varphi_*\boldsymbol{\pi}_n(\gamma) \leq \boldsymbol{\pi}(A_n)^{-1}L^2(B_n(\bar{x}))\iint_0^1 |\dot{\gamma}_t|^2\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}_n(\gamma).$$

Hence $\varphi_*\boldsymbol{\pi}_n$ is a test plan on M_2 and thus for $\varphi_*\boldsymbol{\pi}_n$-a.e. $\tilde{\gamma}$ we have that $t \mapsto f(\tilde{\gamma})$ is in $W^{1,1}(0,1)$ with

$$|\frac{\mathrm{d}}{\mathrm{d}t}f(\tilde{\gamma}_t)| \leq |\dot{\tilde{\gamma}}_t||\mathrm{d}f|(\tilde{\gamma}_t).$$

Recalling (1.3.4.7) this means that for $\boldsymbol{\pi}_n$-a.e. γ the map $t \mapsto f(\varphi(\gamma_t))$ belongs to $W^{1,1}(0,1)$ with

$$|\frac{\mathrm{d}}{\mathrm{d}t}f(\varphi(\gamma_t))| \leq \mathrm{lip}\,\varphi(\gamma_t)\,|\dot{\gamma}_t|\,|\mathrm{d}f|(\varphi(\gamma_t)). \tag{1.3.4.8}$$

Being this true for every $n \in \mathbb{N}$ sufficiently large, (1.3.4.8) holds also for $\boldsymbol{\pi}$-a.e. γ and since $\mathrm{lip}\,\varphi\,|\mathrm{d}f| \circ \varphi \in L^2_{\mathsf{loc}}(\mathrm{M}_1)$, by the characterization (1.3.1.3) of Sobolev functions the proof is completed. □

Theorem/Definition 1.3.20 (Pullback of 1-forms). *Let* $(\mathrm{M}_1, \mathsf{d}_1, \mathfrak{m}_1)$ *and* $(\mathrm{M}_2, \mathsf{d}_2, \mathfrak{m}_2)$ *be two metric measure spaces and* $\varphi : \mathrm{M}_1 \to \mathrm{M}_2$ *be a map of local bounded deformation.*

Then there exists a unique linear and continuous map $\varphi^* \colon L^0(T^*\mathrm{M}_2) \to L^0(T^*\mathrm{M}_1)$ *such that*

$$\begin{aligned} \varphi^*(\mathrm{d}f) &= \mathrm{d}(f \circ \varphi), && \forall f \in \mathcal{S}^2_{\mathsf{loc}}(\mathrm{M}_2),\\ \varphi^*(g\omega) &= g \circ \varphi\,\varphi^*\omega, && \forall g \in L^0(\mathrm{M}_2),\ \omega \in L^0(T^*\mathrm{M}_2), \end{aligned} \tag{1.3.4.9}$$

and such map satisfies

$$|\varphi^*\omega| \leq \mathrm{lip}\,\varphi\,|\omega| \circ \varphi \qquad \mathfrak{m}_1\text{-}a.e., \qquad \forall \omega \in L^0(T^*\mathrm{M}_2). \tag{1.3.4.10}$$

proof The requirements (1.3.4.9) force the definition

$$\varphi^*\omega := \sum_i \chi_{\varphi^{-1}(E_i)}\mathrm{d}(f_i \circ \varphi) \qquad \text{for} \qquad \omega = \sum_i \chi_{E_i}\mathrm{d}f_i, \tag{1.3.4.11}$$

for (E_i) finite Borel partition of M_2 and $(f_i) \subset \mathcal{S}^2_{\mathsf{loc}}(\mathrm{M}_2)$. The bound

$$|\varphi^*\omega| = \sum_i \chi_{\varphi^{-1}(E_i)}|\mathrm{d}(f_i \circ \varphi)| \overset{(1.3.4.6)}{\leq} \mathrm{lip}\,\varphi \sum_i \chi_{E_i} \circ \varphi\,|\mathrm{d}f_i| \circ \varphi = \mathrm{lip}\,\varphi\,|\omega| \circ \varphi, \tag{1.3.4.12}$$

grants both that the definition of $\varphi^*\omega$ is well-posed (i.e. its value depends only on ω and not in how it is written as $\sum_i \chi_{E_i}\mathrm{d}f_i$) and that φ^* is continuous from the space of ω's as in (1.3.4.11) with the $L^0(T^*\mathrm{M}_2)$-topology to $L^0(T^*\mathrm{M}_1)$. Since the class of such ω's is dense in $L^0(T^*\mathrm{M}_2)$, we can be uniquely extend φ^* to a continuous map from $L^0(T^*\mathrm{M}_2)$ to $L^0(T^*\mathrm{M}_1)$.

The resulting extension satisfies the first in (1.3.4.9) by definition, while (1.3.4.10) comes from (1.3.4.12). The second in (1.3.4.9) for simple functions g is a direct consequence of the definition (1.3.4.11), then the general case follows by approximation. □

We remark that the composition of two maps φ, ψ of locally bounded deformation is a map of locally bounded deformation and that

$$(\varphi \circ \psi)^* = \psi^* \circ \varphi^*.$$

It is worth to underline that given a map of locally bounded deformation $\varphi\colon \mathrm{M}_2 \to \mathrm{M}_1$ there are two very different ways of considering the pull-back of 1-forms: the one defined in Theorem 1.3.20, which takes value in $L^0(T^*\mathrm{M}_2)$, and the one in the sense of pull-back modules as introduced in the previous section 1.3.4.1, namely as an element of the pullback module $\varphi^*L^0(T^*\mathrm{M}_1)$ of $L^0(T^*\mathrm{M}_1)$ through φ. Accordingly to the standard notation, we keep the notation φ^* for the just defined pullback, while we denote by $[\varphi^*]$ the one in the sense of pull-back modules.

We can now define by duality the differential of a map of locally bounded deformation:

Theorem/Definition 1.3.21 (Differential of a map of locally bounded deformation)**.** *Let $\varphi\colon \mathrm{M}_2 \to \mathrm{M}_1$ be a map of locally bounded deformation and suppose that $L^0(T\mathrm{M}_1)$ is separable. Then there exists a unique $L^0(\mathfrak{m}_2)$-linear and continuous map $\mathrm{d}\varphi\colon L^0(T\mathrm{M}_2) \to \varphi^*L^0(T\mathrm{M}_1)$, that we call the differential of φ, such that*

$$[\varphi^*\omega](\mathrm{d}\varphi(v)) = \varphi^*\omega(v), \qquad \forall\, \omega \in L^0(T^*\mathrm{M}_1),\ v \in L^0(T\mathrm{M}_2) \tag{1.3.4.13}$$

and it satisfies

$$|\mathrm{d}\varphi(v)| \le \mathrm{Lip}(\varphi)|v| \qquad \mathfrak{m}_2\text{-}a.e. \quad \forall\, v \in L^0(T\mathrm{M}_2) \tag{1.3.4.14}$$

proof For any $v \in L^0(T\mathrm{M}_2)$ we consider the map $L_v\colon \{\varphi^*\omega : \omega \in L^0(T\mathrm{M}_1)\} \to L^0(\mathfrak{m}_2)$ which sends $\varphi^*\omega$ to $\varphi^*\omega(v)$. The bound in (1.3.4.10) and the identity $|\omega| \circ \varphi = |[\varphi^*]\omega|$ give

$$|L_v(\omega)| \le \mathrm{Lip}(\varphi)|[\varphi^*]\omega||v| \qquad \mathfrak{m}_2\text{-a.e.}, \quad \forall \omega \in L^0(T\mathrm{M}_1).$$

The vector space $\{\varphi^*\omega : \omega \in L^0(T^*\mathrm{M}_1)\}$ generates $\varphi^*L^0(T^*\mathrm{M}_1)$; now the separability assumption together with Theorem 1.3.17 ensure that the dual of $\varphi^*L^0(T^*\mathrm{M}_1)$ is given by $\varphi^*L^0(T\mathrm{M}_1)$ and so, by Proposition 1.2.10 there exists a unique element in $\varphi^*L^0(T^*\mathrm{M}_1)$, that we denote by $\mathrm{d}\varphi(v)$, for which (1.3.4.13) holds and (1.3.4.14) is satisfied.

We can then conclude just observing that the map $v \mapsto \mathrm{d}\varphi(v)$ is $L^0(\mathfrak{m})$-linear while the continuity is ensured by the bound (1.3.4.14). □

Remark 1.3.22. In the case in which φ is invertible with inverse of locally bounded deformation, then Remark 1.3.15 grants that the pullback module $\varphi^*L^0(T\mathrm{M})$ can be identified with $L^0(T\mathrm{M})$ via the pullback map. Therefore the differential $\mathrm{d}\varphi$ can be seen as a map from $L^0(T\mathrm{M}_2)$ to $L^0(T\mathrm{M}_1)$ and the identity in (1.3.4.13) takes the form

$$\omega(\mathrm{d}\varphi(v)) = \varphi^*\omega(v) \circ \varphi^{-1}. \tag{1.3.4.15}$$

Remark 1.3.23 (The map φ is of bounded deformation/compression). In the case in which φ is a map of bounded deformation (respectively of bounded compression) all the results found in this section (in Section 1.3.4.1) are still valid and actually we consider L^2-modules instead of L^0-ones and replace the L^0-linearity of the maps with the L^∞ one.

1.3.5 Infinitesimally Hilbertian spaces and Laplacian

In this section we specialize the first order calculus on metric measure spaces which, from the Sobolev calculus point of view, resemble Riemannian manifolds rather than the more general Finsler ones.

Definition 1.3.24 (Infinitesimally Hilbertian spaces)**.** *A metric measure space* $(\mathrm{M}, \mathsf{d}, \mathfrak{m})$ *is said to be infinitesimally Hilbertian provided* $W^{1,2}(\mathrm{M})$ *is an Hilbert space.*

We recall that in [36] it has been proved that the infinitesimally Hilbertianity of the space is actually equivalent to the fact that both $L^2(T^*\mathrm{M})$ and $L^2(T\mathrm{M})$ are Hilbert modules, which are in particular endowed with a pointwise scalar product. An implication is easy to show: indeed since the map $f \mapsto (f, \mathrm{d}f)$ is an isometry of $W^{1,2}(\mathrm{M})$ into $L^2(\mathfrak{m}) \times L^2(T^*\mathrm{M})$ endowed with the norm $\|(f,\omega)\|^2 := \|f\|^2_{L^2(\mathfrak{m})} + \|\omega\|^2_{L^2(\mathfrak{m})}$, it follows that if M is infinitesimally Hilbertian, then $W^{1,2}(\mathrm{M})$ is a Hilbert space. The non trivial implication is the converse one, i.e. if $W^{1,2}(\mathrm{M})$ is Hilbert, then so is $L^2(T^*\mathrm{M})$.

It is worth to recall that the reflexivity of the Sobolev space $W^{1,2}(\mathrm{M})$ in an infinitesimally Hilbertian space implies that $W^{1,2}(\mathrm{M})$ is actually separable (we refer to [3] for a proof of this result). Therefore $L^2(T^*\mathrm{M})$ is separable too, since $L^2(T^*\mathrm{M})$ is generated by differentials.

Definition 1.3.24 underlines a "global" Hilbert property of the space, but actually the characterization above shows that it is equivalent to a "pointwise" Hilbertianity of the space, which justifies the word "infinitesimally" in the terminology.

Therefore if $(\mathrm{M}, \mathsf{d}, \mathfrak{m})$ is infinitesimally Hilbertian, by Proposition 1.2.18 we know that $L^2(T^*\mathrm{M})$ and $L^2(T\mathrm{M})$ are isomorphic as L^∞-modules. For $f \in W^{1,2}(\mathrm{M})$ (or in $W^{1,2}_{\mathsf{loc}}(\mathrm{M})$), the image of $\mathrm{d}f$ under the musical isomorphism (1.3.3.1) is called *gradient* of f and denoted by $\nabla f = (\mathrm{d}f)^\sharp \in L^2(T\mathrm{M})$ (or in $L^2_{\mathsf{loc}}(T\mathrm{M})$). The chain rule (1.3.2.2) and the Leibniz one (1.3.2.3) for the differential guarantee that

$$\begin{aligned} \nabla(\varphi \circ f) &= \varphi' \circ f \; \nabla f, && \forall\, f \in \mathcal{S}^2(\mathrm{M}),\ \varphi \in \mathrm{Lip} \cap C^1(\mathbb{R}),\\ \nabla(fg) &= f\nabla g + g\nabla f, && \forall\, f,g \in L^\infty \cap \mathcal{S}^2(\mathrm{M}). \end{aligned}$$

Moreover in Section 4.3 of [9] it has been proved that in an infinitesimally Hilbertian space for every $f, g \in \mathcal{S}^2(\mathrm{M})$ it holds

$$\mathrm{d}f(\nabla g) = \mathrm{d}g(\nabla f), \qquad \mathfrak{m}\text{-a.e.} \tag{1.3.5.1}$$

Definition 1.3.25 (Laplacian). *The space* $\mathrm{D}(\Delta)$ *(resp.* $\mathrm{D}(\Delta_{\mathsf{loc}})$*) is the space of all functions* $f \in W^{1,2}(\mathrm{M})$ *(resp.* $f \in W^{1,2}_{\mathsf{loc}}(\mathrm{M})$*) such that there exists* $h \in L^2(\mathfrak{m})$ *(in* $L^2_{\mathsf{loc}}(\mathfrak{m})$*) for which*

$$\int hg \,\mathrm{d}\mathfrak{m} = -\int \langle \nabla f, \nabla g\rangle \,\mathrm{d}\mathfrak{m}, \qquad \forall g \in W^{1,2}(\mathrm{M}).$$

In this case the function h *is called Laplacian of* f *and denote by* Δf.

This means that Δ is the infinitesimal generator associated to the Dirichlet form $\mathsf{E}\colon L^2(\mathfrak{m}) \to [0, +\infty]$, which we call *Cheeger energy*, given by

$$\mathsf{E}(f) := \begin{cases} \dfrac{1}{2}\displaystyle\int |\mathrm{D}f|^2 \,\mathrm{d}\mathfrak{m}, & \text{if } f \in W^{1,2}(\mathrm{M}),\\ +\infty, & \text{otherwise.} \end{cases} \tag{1.3.5.2}$$

Directly from the properties of the minimal weak upper gradient we deduce that E is convex and lower semicontinuous; in particular it is a closed operator with dense domain, i.e., $\{f : \mathsf{E}(f) < \infty\}$ is dense in $L^2(\mathfrak{m})$. This ensures that $\mathrm{D}(\Delta)$ is dense in $W^{1,2}(\mathrm{M})$ and from the definition we have

$$f \in \mathrm{D}(\Delta) \quad \Longleftrightarrow \quad \nabla f \in \mathrm{D}(\mathrm{div}) \text{ and in this case } \Delta f = \mathrm{div}(\nabla f),$$

and so, recalling (1.3.3.5), it follows

$$\text{on infinitesimally Hilbertian spaces the space } \mathrm{D}(\mathrm{div}) \text{ is dense in } L^2(T\mathrm{M}). \tag{1.3.5.3}$$

Moreover, directly from the calculus rules for the differential, a direct computation shows the following properties:

$$\Delta(\varphi \circ f) = \varphi' \circ f \ \Delta f + \varphi'' \circ f \ |\nabla f|^2, \qquad \forall\ f \in \mathrm{Lip}_b(\mathrm{M}) \cap \mathrm{D}(\Delta), \varphi \in C^2(\mathbb{R}), \tag{1.3.5.4}$$

$$\Delta(fg) = f\Delta g + g\Delta f + 2\langle \nabla f, \nabla g\rangle, \qquad \forall\ f, g \in \mathrm{Lip}_b(\mathrm{M}) \cap \mathrm{D}(\Delta). \tag{1.3.5.5}$$

We conclude this section with the following:

Theorem 1.3.26 (Derivation along geodesics)**.** *Let* $(\mathrm{M}, \mathsf{d}, \mathfrak{m})$ *be an infinitesimally Hilbertian space and* $t \mapsto \mu_t = \rho_t \mathfrak{m} \in \mathscr{P}_2(\mathrm{M})$ *be a geodesics made of measures with uniformly bounded supports and densities. Let us assume that the map* $t \mapsto \rho_t \in L^p(\mathfrak{m})$ *is continuous for some (and thus for any)* $p \in [1, \infty)$.

Then for every $f \in W^{1,2}(\mathrm{M})$ *the map* $t \mapsto \int f \ \mathrm{d}\mu_t$ *is* $C^1([0,1])$ *and it holds*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int f \ \mathrm{d}\mu_t = -\int \langle \nabla f, \nabla \varphi_t\rangle \ \mathrm{d}\mu_t, \qquad \forall\ t \in [0,1] \tag{1.3.5.6}$$

where for every $t \in [0,1]$ *the function* φ_t *is Lipschitz and such that there exists* $s \in [0,1]$, $s \neq t$, *with the property that the function* $(s-t)\varphi$ *is a Kantorovich potential from* μ_t *to* μ_s.

Remark 1.3.27. We refer to [61] for a proof of the fact that on a $\mathsf{RCD}(K, \infty)$ space every W_2-geodesic between two measures μ_0 and μ_1 with bounded supports and densities satisfies the hypothesis of Theorem 1.3.26.

1.4 Second order differential theory for RCD spaces

1.4.1 Definition and basic properties of RCD spaces

Let $(\mathrm{M}, \mathsf{d}, \mathfrak{m})$ be a complete and separable metric space endowed with a non-negative Radon measure. We start with the definition of weak Ricci curvature bound, with any condition on the dimension. In order to do it we define the *relative entropy* functional $\mathrm{Ent}_\mathfrak{m} \colon \mathscr{P}(\mathrm{M}) \to \mathbb{R} \cup \{+\infty\}$ as

$$\mathrm{Ent}_\mathfrak{m}(\mu) := \begin{cases} \displaystyle\int \rho \ \log(\rho) \ \mathrm{d}\mathfrak{m}, & \text{if } \ \mu = \rho\mathfrak{m} \text{ and } (\rho \ \log(\rho))^- \in L^1(\mathfrak{m}), \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.4.1.1}$$

Definition 1.4.1 ($\mathsf{CD}(K,\infty)$ and $\mathsf{RCD}(K,\infty)$ spaces)**.** *Given* $K \in \mathbb{R}$ *we say that* $(\mathrm{M}, \mathsf{d}, \mathfrak{m})$ *is a* $\mathsf{CD}(K,\infty)$ *space if the functional* $Ent_\mathfrak{m}$ *is* K*-geodesically convex in* $\mathscr{P}_2(\mathrm{M})$, W_2, *namely if for every* $\mu, \nu \in \mathscr{P}_2(\mathrm{M})$ *with* $Ent_\mathfrak{m}(\mu), Ent_\mathfrak{m}(\nu) < \infty$ *there exists a* W_2*-geodesics* (μ_t) *with* $\mu_0 = \mu$ *and* $\mu_1 = \nu$ *and such that*

$$Ent_\mathfrak{m}(\mu_t) \leq (1-t)Ent_\mathfrak{m}(\mu) + tEnt_\mathfrak{m}(\nu) - \frac{K}{2}t(1-t)\, W_2^2(\mu,\nu), \quad \forall t \in [0,1].$$

We say that $(\mathrm{M}, \mathsf{d}, \mathfrak{m})$ *is an* $\mathsf{RCD}(K,\infty)$ *space if it is an infinitesimally Hilbertian* $\mathsf{CD}(K,\infty)$ *space.*

On $\mathsf{CD}(K,\infty)$ spaces, there exists a constant $C > 0$ such that for any $x \in \mathrm{M}$ it holds

$$\mathfrak{m}(B_r(x)) \leq Ce^{Cr^2}, \qquad \forall\ r > 0$$

(see [67] for a proof of this fact). This bound ensures that for $\mu \in \mathscr{P}_2(\mathrm{M})$ with $\mu = \rho\mathfrak{m}$ we always have that $(\rho \log(\rho))^- \in L^1(\mathfrak{m})$ and from this fact it follows that $\mathrm{Ent}_\mathfrak{m}$ is lower semicontinuous on $(\mathscr{P}_2(\mathrm{M}), W_2)$.

We refer to [9] and to [35] for a proof of the following important consequence of the Ricci curvature lower bound, which allows to pass from a Sobolev information to a metric one:

Theorem 1.4.2 (Sobolev-to-Lipschitz property). *Let* $(\mathrm{M}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K, \infty)$ *space. Then every* $f \in W^{1,2}(\mathrm{M})$ *with* $|\mathrm{D}f| \in L^\infty(\mathrm{M})$*. Then there exists* $\tilde{f} = f$ $\mathfrak{m}$*-a.e. such that* $\mathrm{Lip}(\tilde{f}) \leq \||\mathrm{D}f|\|_{L^\infty}$.

A relevant property of Sobolev functions in $\mathsf{RCD}(K, \infty)$ spaces in relation to the metric of such spaces is the following result, proved in [9] (see also [35], [37] for the given formulation):

Theorem 1.4.3. *Let* $(\mathrm{M}_1, \mathsf{d}_1, \mathfrak{m}_1)$ *and* $(\mathrm{M}_2, \mathsf{d}_2, \mathfrak{m}_2)$ *be two* $\mathsf{RCD}(K, \infty)$ *spaces with* $\mathfrak{m}_1, \mathfrak{m}_2$ *having full support and* $T : \mathrm{M}_1 \to \mathrm{M}_2$ *and* $S : \mathrm{M}_2 \to \mathrm{M}_1$ *be Borel maps such that*

$$T \circ S = \mathrm{Id}_{\mathrm{M}_2} \quad \mathfrak{m}_2 - a.e. \qquad\qquad S \circ T = \mathrm{Id}_{\mathrm{M}_1} \quad \mathfrak{m}_1 - a.e.,$$

and

$$T_* \mathfrak{m}_1 = \mathfrak{m}_2 \qquad\qquad \mathsf{E}_{\mathrm{M}_1}(f \circ T) = \mathsf{E}_{\mathrm{M}_2}(f) \quad \forall f \in L^2(\mathrm{M}_2).$$

Then, up to modifications in a $\mathfrak{m}_1$*-negligible set,* T *is an isometry.*

It is useful to have also a notion of weak Ricci curvature bound with also a bound on the dimension. To that purpose we introduce the dimension-dependent entropy functional $\mathcal{U}_N \colon \mathscr{P}(\mathrm{M}) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\mathcal{U}_N := \int -(\rho^{1-\frac{1}{N}})\, \mathrm{d}\mathfrak{m}, \qquad \text{for} \ \ \mu = \rho\mathfrak{m} + \mu_s, \ \ \mu_s \perp \mathfrak{m}.$$

Note that if we take the limit as $N \to \infty$ we find the relative entropy functional defined in (1.4.1.1).

Then for $K \in \mathbb{R}$ and $N \in [1, \infty)$ we introduce the distortion coefficients

$$\sigma^{(t)}_{K,N}(\theta) := \begin{cases} +\infty & \text{if } K\theta^2 \geq N\pi^2, \\ \dfrac{\sin(t\theta\sqrt{K/N})}{\sin(\theta\sqrt{K/N})} & \text{if } 0 < K\theta^2 < N\pi^2, \\ t & \text{if } K\theta^2 = 0, \\ \dfrac{\sin(t\theta\sqrt{-K/N})}{\sin(\theta\sqrt{-K/N})} & \text{if} K\theta^2 < 0. \end{cases} \tag{1.4.1.2}$$

Definition 1.4.4 ($\mathsf{CD}^*(K, N)$ and $\mathsf{RCD}^*(K, N)$ spaces). *For* $K \in \mathbb{R}$ *and* $N \in [1, \infty)$ *we say that* $(\mathrm{M}, \mathsf{d}, \mathfrak{m})$ *is a* $\mathsf{CD}^*(K, N)$ *space if for every* $\mu_i = \rho_i \mathfrak{m} \in \mathscr{P}(\mathrm{M})$, $i = 0, 1$ *with bounded support there exists a measure* $\boldsymbol{\pi} \in OptGeo(\mu_0, \mu_1)$ *such that for every* $t \in [0, 1]$ *and* $N' \geq N$ *it holds*

$$\mathcal{U}_{N'}(\mu_t) \leq \int \Big(\sigma^{(1-s)}_{K/N'}(\mathsf{d}(\gamma_0, \gamma_1)) \rho_0(\gamma_0)^{-1/N'} + \sigma^{(s)}_{K/N'}(\mathsf{d}(\gamma_0, \gamma_1)) \rho_1(\gamma_1)^{-1/N'} \Big)\, \mathrm{d}\boldsymbol{\pi}(\gamma).$$

We say that $(\mathrm{M}, \mathsf{d}, \mathfrak{m})$ *is an* $\mathsf{RCD}^*(K, N)$ *space if it is an infinitesimally Hilbertian* $\mathsf{CD}^*(K, N)$ *space.*

In particular we specialize this definition in the case in which $K = 0$:

Definition 1.4.5 ($\mathsf{CD}^*(0, N)$ and $\mathsf{RCD}^*(0, N)$ spaces). *We say that* $(\mathrm{M}, \mathsf{d}, \mathfrak{m})$ *is a* $\mathsf{CD}^*(0, N)$ *space if the functional* $\mathcal{U}_N$ *is geodesically convex in* $\mathscr{P}_2(\mathrm{M})$, W_2*, namely if for every* $\mu, \nu \in \mathscr{P}_2(\mathrm{M})$ *with* $\mathcal{U}_N(\mu), \mathcal{U}_N(\nu) < \infty$ *there exists a* W_2*-geodesics* (μ_t) *with* $\mu_0 = \mu$ *and* $\mu_1 = \nu$ *and such that*

$$\mathcal{U}_N(\mu_t) \leq (1-t)\, \mathcal{U}_N(\mu) + t\, \mathcal{U}_N(\nu), \qquad \forall t \in [0,1].$$

We say that $(\mathrm{M}, \mathsf{d}, \mathfrak{m})$ *is an* $\mathsf{RCD}^*(0,N)$ *space if it is an infinitesimally Hilbertian* $\mathsf{CD}^*(0,N)$ *space.*

1.4.2 Heat Flow

We recall that in Section 1.3.5 we have defined the Laplacian has the infinitesimal generator associated to the Cheeger energy defined in (1.3.5.2).

The classical theory of gradient flows of convex functions on Hilbert spaces (see [6] and the references therein) ensures the existence and the uniqueness of a 1-parameter semigroup $(\mathsf{h}_t)_{t\geq 0}$ of continuous operators from $L^2(\mathfrak{m})$ to itself such that for every $f \in L^2(\mathfrak{m})$ the curve $t \mapsto \mathsf{h}_t(f)$ is continuous on $[0,\infty)$, locally absolutely continuous on $(0,\infty)$, $\mathsf{h}_t(f) \in \mathrm{D}(\Delta)$ for any $t>0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{h}_t(f) = \Delta f \quad \text{a.e.}\, t>0.$$

We point out that RCD spaces are in particular infinitesimally Hilbertian and so the energy functional E is a quadratic form, the domain of the Laplacian $\mathrm{D}(\Delta)$ is a vector space and the heat flow $(\mathsf{h}_t)_{t\geq 0}$ is linear and self-adjoint.

We recall then the following useful estimates, true for every $f \in L^2(\mathfrak{m})$ and for any $t>0$

$$\mathsf{E}(\mathsf{h}_t f) \leq \frac{1}{4t}\, \|f\|^2_{L^2(\mathfrak{m})}, \tag{1.4.2.1}$$

$$\|\Delta \mathsf{h}_t f\|^2_{L^2(\mathfrak{m})} \leq \frac{1}{2t^2}\, \|f\|^2_{L^2(\mathfrak{m})}. \tag{1.4.2.2}$$

These two inequalities can be obtained by differentiating the $L^2(\mathfrak{m})$ norm and the energy along the flow. Moreover for every $p \in [1,\infty]$ and for every $f \in L^2 \cap L^p(\mathfrak{m})$ it holds

$$\|\mathsf{h}_t f\|_{L^p(\mathfrak{m})} \leq \|f\|_{L^p(\mathfrak{m})}, \qquad \forall t \geq 0. \tag{1.4.2.3}$$

This in particular implies that the heat flow can be extended by density to a family of linear and continuous functionals $\mathsf{h}_t \colon L^p(\mathfrak{m}) \to L^p(\mathfrak{m})$ of norm bounded by 1 for any $p<\infty$.

In addition an L^p-version of (1.4.2.2) for $p \in (1,\infty)$ holds:

$$\|\Delta \mathsf{h}_t f\|_p \leq \frac{c_p^\Delta}{t}\|f\|_p, \tag{1.4.2.4}$$

for every $f \in L^p \cap L^2(\mathfrak{m})$ and every $t \in (0,1)$. This can be obtained as a consequence of the fact that the heat flow is analytic [66, Thm. III.1] and actually it is equivalent to it, see [76, Sec. IX.10].

Corollary 1.4.6. *Let* $p \in (1,\infty)$ *and let* c_p^Δ *be the constant* (1.4.2.4)*. Then*

$$\|\mathsf{h}_t f - \mathsf{h}_{t-t'} f\|_p \leq \min\left\{ c_p^\Delta \log\left(1 + \frac{t'}{t-t'}\right), 2\right\} \|f\|_p, \qquad \forall f \in L^p \cap L^2(\mathfrak{m})$$

for every $t, t' \in (0,1)$*, with* $t' \leq t$*.*

proof If $p = 2$ the thesis follows from L^p-contractivity of the heat flow in (1.4.2.3). If $p \neq 2$ we use (1.4.2.4) in order to bound

$$\begin{aligned}\|\mathsf{h}_t f - \mathsf{h}_{t-t'} f\|_p &\leq \int_0^{t'} \|\partial_r h_{t-t'+rf}\|_p \,\mathrm{d}r = \int_0^{t'} \|\Delta h_{t-t'+rf}\|_p \,\mathrm{d}r \\ &\leq \int_0^{t'} \frac{c_p^\Delta}{t-t'+r} \,\mathrm{d}r \|f\|_p = c_p^\Delta \log\left(1 + \frac{t'}{t-t'}\right) \|f\|_p\end{aligned}$$

which gives the thesis. □

A very important property of the heat flow, which is strongly linked to the lower curvature bound, is the Bakry-Émery contraction estimate (see [41] and [9])

$$|\nabla \mathsf{h}_t f|^2 \leq e^{-2Kt} \mathsf{h}_t(|\nabla f|^2), \qquad \mathfrak{m}\text{-a.e., } \forall t \geq 0,\ \forall f \in W^{1,2}(\mathrm{M}). \tag{1.4.2.5}$$

Above all, this implies a sort of reverse Poincaré inequality, namely for all $t > 0$ and $f \in L^2(\mathfrak{m})$ we have

$$2I_{2K}(t)|\nabla \mathsf{h}_t f|^2 \leq \mathsf{h}_t f^2 - (\mathsf{h}_t f)^2 \qquad \mathfrak{m}\text{-a.e. in M}, \tag{1.4.2.6}$$

where I_K denotes the real function defined, for an arbitrary $K \in \mathbb{R}$, by

$$I_K(t) := \int_0^t e^{Kr} \,\mathrm{d}r = \begin{cases} \frac{1}{K}\left(e^{Kt} - 1\right) & \text{if } K \neq 0, \\ t & \text{if } K = 0. \end{cases}$$

We refer to [10, Corollary 2.3] for a proof of this result, which in turn implies the following crucial inequality:

$$\|\mathrm{d}\mathsf{h}_t f\|_{L^p(\mathfrak{m})} \leq \frac{c_p}{\sqrt{t}} \|f\|_{L^p(\mathfrak{m})}, \quad \text{for every } f \in L^2 \cap L^p(\mathfrak{m}), t \in (0,1), \tag{1.4.2.7}$$

where $p \in [1, \infty]$ and $c_p > 0$ is a positive constant just depending on p. Indeed it suffices to integrate (1.4.2.6) to obtain

$$(2I_{2K}(t))^{p/2} \int |\nabla \mathsf{h}_t f|^p \,\mathrm{d}\mathfrak{m} \leq \int (\mathsf{h}_t f^2)^{p/2} \,\mathrm{d}\mathfrak{m} \leq \int f^p \,\mathrm{d}\mathfrak{m}$$

and then we can conclude just observing that $2I_{2K}(t)^{-1} = O(t^{-1})$ as $t \downarrow 0$.

1.4.3 Measure valued Laplacian and test functions

In this section we introduce a key tool that we shall use in several instances to develop the second order calculus on RCD spaces, that is the set of *test functions*, $\mathrm{Test}F(\mathrm{M}) \subset W^{1,2}(\mathrm{M})$, first introduced in [64]:

$$\mathrm{Test}F(\mathrm{M}) := \left\{ \mathrm{f} \in \mathrm{D}(\Delta) \cap L^\infty(\mathfrak{m}) : |\nabla \mathrm{f}| \in L^\infty(\mathfrak{m}) \text{ and } \Delta \mathrm{f} \in W^{1,2}(\mathrm{M}) \right\}.$$

In particular test functions form an algebra and the following useful approximation result proved in [13] holds

$$\begin{aligned}&\text{for every } K \subset \Omega \subset \mathrm{M} \text{ with } \mathsf{d}(x,y) \geq c \text{ for some } c > 0 \text{ and every } x \in K,\ y \in \Omega^c \\ &\text{there exists } f \in \mathrm{Test}(\mathrm{M}) \text{ with } \mathrm{supp}(f) \subset \Omega \text{ and identically 1 on } K.\end{aligned} \tag{1.4.3.1}$$

Moreover notice that the Sobolev-to-Lipschitz property in Theorem 1.4.2 ensures that any $f \in \mathrm{Test}F(\mathrm{M})$ has a Lipschitz representative $\bar{f}\colon \mathrm{M} \to \mathbb{R}$ with $\mathrm{Lip}(\bar{f}) \leq \||\nabla f|\|_{L^\infty(\mathfrak{m})}$, while the $L^\infty \to \mathrm{Lip}$ regularization of the heat flow established in [9] as a direct consequence of the Bakry-Émery contraction estimate grants that

$$f \in L^2 \cap L^\infty(\mathfrak{m}), f \geq 0 \quad \Rightarrow \quad \mathsf{h}_t f \in \mathrm{Test}F(\mathrm{M}),\ \mathsf{h}_t\mathrm{f} \geq 0, \quad \forall\ \mathrm{t} > 0, \tag{1.4.3.2}$$

which in particular implies that

$$\mathrm{Test}F(\mathrm{M}) \text{ is dense in } W^{1,2}(\mathrm{M}). \tag{1.4.3.3}$$

At this point, in order to further analyze the regularity properties of functions in $\mathrm{Test}F(\mathrm{M})$, we introduce the notion of measure valued Laplacian, which arises naturally from integration by parts (we refer to [38]).

Definition 1.4.7 (Measure valued Laplacian)**.** *Let $f \in W^{1,2}(\mathrm{M})$. We say that f has a measure valued Laplacian, and write $f \in \mathrm{D}(\mathbf{\Delta})$, if there exists a Borel measure μ on* M *finite on bounded sets such that*

$$\int g \,\mathrm{d}\mathfrak{m} = -\int \langle \nabla f, \nabla g\rangle \,\mathrm{d}\mathfrak{m} \quad \textit{for every } g \in \mathrm{Lip}(\mathrm{M}) \textit{ with bounded support.}$$

In this case the measure μ, which is unique, is denoted by $\mathbf{\Delta} f$.

The bilinearity of $(f, g) \mapsto \langle \nabla f, \nabla g\rangle \in L^1(\mathfrak{m})$ ensures that $\mathrm{D}(\mathbf{\Delta})$ is a vector space and that the map $\mathbf{\Delta}\colon \mathrm{D}(\mathbf{\Delta}) \to \mathrm{Meas}(\mathrm{M})$ is linear, where the Banach space Meas(M) of finite Radon measures on M equipped with the total variation norm $\|\cdot\|_{\mathrm{TV}}$.

It is worth to underline that this notion is fully compatible with the one of Laplacian given in Definition 1.3.25, in the sense that

$$f \in \mathrm{D}(\Delta) \Leftrightarrow f \in \mathrm{D}(\mathbf{\Delta}) \text{ with } \mathbf{\Delta} f \ll \mathfrak{m} \text{ and } \frac{\mathrm{d}\mathbf{\Delta} f}{\mathrm{d}\mathfrak{m}} \in L^2(\mathfrak{m}), \text{ and in this case } \mathbf{\Delta} f = \Delta f \mathfrak{m}.$$

The following result is crucial to develop the second order calculus in RCD spaces, since above other things it provides Sobolev regularity for $|\mathrm{d}f|^2$ for any $f \in \mathrm{Test}F(\mathrm{M})$. For a proof of this Theorem we refer to [64], where the arguments proposed in [18] have been generalized and adapted to this context (see also [21]). In particular we remark that without any lower Ricci bound it is not possible by now to exhibit a non-constant function f for which $|\nabla f|$ has any kind of regularity.

Theorem 1.4.8. *Let $f \in \mathrm{Test}F(\mathrm{M})$. Then $|\nabla f|^2 \in \mathrm{D}(\mathbf{\Delta}) \subset W^{1,2}(\mathrm{M})$ and*

$$\begin{aligned} &\mathsf{E}(|\nabla f|^2) \leq -\int K|\nabla f|^4 + |\nabla f|\langle \nabla f, \nabla \Delta f\rangle \,\mathrm{d}\mathfrak{m}, \\ &\frac{1}{2}\mathbf{\Delta}|\nabla f|^2 \geq (K|\nabla f|^2 + \langle \nabla f, \nabla \Delta f\rangle)\mathfrak{m} \end{aligned} \tag{1.4.3.4}$$

Finally we notice that for any $f, g \in \mathrm{Test}F(\mathrm{M})$, we have $fg \in L^\infty \cap W^{1,2}(\mathrm{M})$ with $|\nabla(fg)| \in L^\infty(\mathfrak{m})$ and the Leibniz rule for the Laplacian in (1.3.5.5) ensures that $fg \in \mathrm{D}(\Delta)$ with

$$\Delta(fg) = f\Delta g + g\Delta f + 2\langle \nabla f, \nabla g\rangle.$$

The fact that $\Delta f, \Delta g$ are in $W^{1,2}(\mathrm{M})$ and that f, g are bounded with bounded differential imply that both $f\Delta g$ and $g\Delta f$ are in $W^{1,2}(\mathrm{M})$. Hence by Theorem 1.4.8 and polarization we have that $\langle \nabla f, \nabla g\rangle \in W^{1,2}(\mathrm{M})$ and in particular this proves that

$$\mathrm{Test}F(\mathrm{M}) \text{ is an algebra.} \tag{1.4.3.5}$$

In [31] it has been proved that on $\mathsf{RCD}^*(0,N)$ spaces the *Bochner inequality* holds in the sense that for $f \in \mathrm{Test}F(\mathrm{M})$ the function $|\mathrm{D}f|^2$ has a measure valued Laplacian and

$$\mathbf{\Delta}\frac{|\mathrm{D}f|^2}{2} \geq \left(\frac{(\Delta f)^2}{N} + \langle \nabla f, \nabla \Delta f \rangle\right)\mathfrak{m}. \tag{1.4.3.6}$$

1.4.4 The space $W^{2,2}(\mathrm{M})$ and the Hessian

In this section we write $L^2((T^*)^{\otimes 2}\mathrm{M})$ and $L^2(T^{\otimes 2}\mathrm{M})$ for respectively the tensor product of $L^2(T^*\mathrm{M})$ and $L^2(T\mathrm{M})$ with itself (see Section 1.2.5). These modules are one the dual of the other and we shall write $A(X,Y)$ in place of $A(X\otimes Y)$ for $A \in L^2((T^*)^{\otimes 2}\mathrm{M})$ and $X \otimes Y \in L^2(T^{\otimes 2}\mathrm{M})$.

Recall that, being respectively $L^2(T^*\mathrm{M})$ and $L^2(T\mathrm{M})$ separable (Remark 1.3.4), also the modules $L^2((T^*)^{\otimes 2}\mathrm{M})$ and $L^2(T^{\otimes 2}\mathrm{M})$ are separable.

We start recalling that on a smooth Riemannian manifold the Hessian of a smooth function f is characterized by the validity of the identity

$$2\mathrm{Hess}(f)(\nabla g_1, \nabla g_2) = \langle \nabla g_1, \nabla\langle \nabla f, \nabla g_2\rangle\rangle + \langle \nabla g_2, \nabla\langle \nabla f, \nabla g_1\rangle\rangle - \langle \nabla f, \nabla\langle \nabla g_1, \nabla g_2\rangle\rangle$$

for a sufficiently large class of test functions g_1, g_2. Multiplying both sides of this identity by a smooth function h and then integrating we get

$$\begin{aligned}2\int & h\mathrm{Hess}(f)(\nabla g_1,\nabla g_2)\,\mathrm{d}\mathfrak{m}\\ &= \int -\langle \nabla f, \nabla g_2\rangle \mathrm{div}(h\nabla g_1) - \langle \nabla f, \nabla g_1\rangle \mathrm{div}(h\nabla g_2) - h\langle \nabla f, \nabla\langle \nabla g_1, \nabla g_2\rangle\rangle\,\mathrm{d}\mathfrak{m},\end{aligned}$$

and the validity of this identity for a sufficiently large class of test functions h, g_1, g_2 still characterizes the Hessian of f. Actually this second formulation has the advantage that in the right hand side only one derivative of f appears, so that we can define which are the functions having an Hessian using this identity.

Notice that for $h, g_1, g_2 \in \mathrm{Test}F(\mathrm{M})$ it holds $\mathrm{div}(h\nabla g_1) = \langle \nabla h, \nabla g\rangle + h\Delta g \in L^2(\mathfrak{m})$ and so $\nabla g_2 \mathrm{div}(h\nabla g_1) \in L^2(T\mathrm{M})$, while Theorem 1.4.8 ensures that also $h\nabla\langle \nabla g_1, \nabla g_2\rangle \in L^2(T\mathrm{M})$.

Definition 1.4.9 (The space $W^{2,2}(\mathrm{M})$ (resp. $W^{2,2}_{\mathsf{loc}}(\mathrm{M})$) and the Hessian)**.** *The space* $W^{2,2}(\mathrm{M}) \subset W^{1,2}(\mathrm{M})$ *(resp.* $W^{2,2}_{\mathsf{loc}}(\mathrm{M}) \subset W^{1,2}_{\mathsf{loc}}(\mathrm{M})$*) is the space of all functions* $f \in W^{1,2}(\mathrm{M})$ *for which there exists* $A \in L^2((T^*)^{\otimes 2}\mathrm{M})$ *(resp. in* $L^2_{\mathsf{loc}}((T^*)^{\otimes 2}\mathrm{M})$*) such that for any* $h, g_1, g_2 \in \mathrm{Test}F(\mathrm{M})$ *(resp. for any* $h, g_1, g_2 \in \mathrm{Test}F(\mathrm{M})$ *with bounded support) the equality*

$$\begin{aligned}2\int & hA(\nabla g_1,\nabla g_2)\,\mathrm{d}\mathfrak{m}\\ &= \int -\langle \nabla f, \nabla g_2\rangle \mathrm{div}(h\nabla g_1) - \langle \nabla f, \nabla g_1\rangle \mathrm{div}(h\nabla g_2) - h\langle \nabla f, \nabla\langle \nabla g_1, \nabla g_2\rangle\rangle\,\mathrm{d}\mathfrak{m},\end{aligned} \tag{1.4.4.1}$$

holds. In this case the operator A *will be called* Hessian of f *and denoted as* $\mathrm{Hess}(f)$.

We endow $W^{2,2}(\mathrm{M})$ *with the norm*

$$\|f\|^2_{W^{2,2}(\mathrm{M})} := \|f\|^2_{L^2(\mathfrak{m})} + \|\mathrm{d}f\|^2_{L^2(T^*\mathrm{M})} + \|\mathrm{Hess}(f)\|^2_{L^2((T^*)^{\otimes 2}\mathrm{M})}.$$

We remark that the density of test functions in $W^{1,2}(\mathrm{M})$ together with the fact that $\mathrm{Test}F(\mathrm{M})$ is an algebra ensure that there exists at most one $A \in L^2((T^*)^{\otimes 2}\mathrm{M})$ for which (1.4.4.1) holds, so that the Hessian, if it exists, is uniquely defined. Thus it linearly depends on f, so that $W^{2,2}(\mathrm{M})$ is a vector space and $\|\cdot\|_{W^{2,2}(\mathrm{M})}$ is a norm.

In the following theorem we collect the basic properties of this space:

Theorem 1.4.10. *We have:*

i) $W^{2,2}(\mathrm{M})$ *is a separable Hilbert space;*

ii) The Hessian is a closed operator, namely the set $\{(f, \mathrm{Hess}(f)) : f \in W^{2,2}(\mathrm{M})\}$ *is a closed subset of* $W^{1,2}(\mathrm{M}) \times L^2((T^*)^{\otimes 2}\mathrm{M})$*;*

iii) For every $f \in W^{2,2}(\mathrm{M})$ *the Hessian* $\mathrm{Hess}(f)$ *is symmetric, i.e.* $\mathrm{Hess}(f)^t = \mathrm{Hess}(f)$.

proof For given $h, g_1, g_2 \in \mathrm{Test}F(\mathrm{M})$ the left hand side in (1.4.4.1) is continuous with respect to $A \in L^2((T^*)^{\otimes 2}\mathrm{M})$, while the right hand side is continuous with respect to $f \in W^{1,2}(\mathrm{M})$. This shows point (*ii*) and the completeness of $W^{2,2}(\mathrm{M})$. Let us prove point (*i*): the fact that the $W^{2,2}$-norm satisfies the parallelogram rule follows by definition, while to prove the separability of $W^{2,2}(\mathrm{M})$ we observe that the space $L^2(\mathfrak{m}) \times L^2(T^*\mathrm{M}) \times L^2((T^*)^{\otimes 2}\mathrm{M})$ endowed with its natural Hilbert structure is separable and that the map

$$W^{2,2}(\mathrm{M}) \ni f \;\mapsto\; (f, \mathrm{d}f, \mathrm{Hess}(f)) \in L^2(\mathfrak{m}) \times L^2(T^*\mathrm{M}) \times L^2((T^*)^{\otimes 2}\mathrm{M})$$

is an isometry. Point (*iii*) follows by the symmetry in g_1 and g_2 of (1.4.4.1). □

Remark 1.4.11 (Hessian and Laplacian). In general the Laplacian is not the trace of the Hessian. The reason of that is given by the fact that the Laplacian is defined via integration by parts, and so it takes into account the reference measure, while the Hessian is a purely differential object.

An example is given by weighted Riemannian manifolds: let M be a Riemannian manifold equipped with a weighted measure $\mathfrak{m} := e^{-V}\mathrm{vol}$ for some smooth function V: in this case the Hessian of a smooth function is independent on the choice of V, while the Laplacian is given by

$$\Delta_{\mathfrak{m}} f = \Delta f - \langle \nabla f, \nabla V \rangle$$

and so it links the Laplacian Δ on the unweighted manifold (i.e., $\int g \Delta f \,\mathrm{dvol} = -\int \langle \nabla g, \nabla f \rangle \,\mathrm{dvol}$) to the Laplacian $\Delta_{\mathfrak{m}}$ on the weighted one (i.e. $\int g \Delta f \,\mathrm{d}\mathfrak{m} = -\int \langle \nabla g, \nabla f \rangle \,\mathrm{d}\mathfrak{m}$).

1.4.4.1 Existence of $W^{2,2}$ functions

At this point we don't know whether the space $W^{2,2}(\mathrm{M})$ contains any non-constant function. In particular it is not even obvious that test functions are actually in $W^{2,2}(\mathrm{M})$. The key result that, among other things, ensures a positive answer to these questions is the following Lemma which is about a self-improvement of Bochner inequality. In the following we are going to show the main outlines of it and to introduce some of the outcoming results, referring to [36] for a detailed proof of all of them.

Read in the setting of a smooth Riemannian manifold with Ricci curvature bounded from below by K the lemma says that for a vector field X and a 2-tensor field A it holds

$$|\nabla X : A|^2 \leq \left(\Delta \frac{|X|^2}{2} + \langle X, (\Delta_H X^\flat)^\sharp \rangle - K|X|^2 - |(\nabla X)_{\mathrm{Asym}}|^2_{\mathrm{HS}} \right) |A|^2_{\mathrm{HS}}. \tag{1.4.4.2}$$

where ∇X is the covariant derivative of X, $(\nabla X)_{\mathrm{Asym}}$ is its antisymmetric part, and $\Delta_H X$ the Hodge Laplacian. Since we haven't introduced these operators yet, we want to restate (1.4.4.2) implicitly for a test vector field $X := \sum_i g_i \, \nabla f_i$ and a test 2-tensor field $A := \sum_j \nabla h_j \otimes \nabla h_j$, for $f_i, g_i, h_i \in \mathrm{Test}F(\mathrm{M})$.

The fact that $\langle \nabla f, \nabla g\rangle \in W^{1,2}(\mathrm{M})$ for every $f, g \in \mathrm{Test}F(\mathrm{M})$ allows to introduce a sort of "Hessian" $H[f]\colon [\mathrm{Test}F(\mathrm{M})]^2 \to L^1(\mathfrak{m})$ of a function $f \in \mathrm{Test}F(\mathrm{M})$ as

$$H[f](g,h) := \frac{1}{2}\big(\langle \nabla(\langle \nabla f, \nabla g\rangle), \nabla h\rangle + \langle \nabla(\langle \nabla f, \nabla h\rangle), \nabla g\rangle - \langle \nabla f, \nabla(\langle \nabla g, \nabla h\rangle)\rangle\big),$$

the terminology being justified by the fact that, as we have seen at the beginning of this section, in the smooth case $H[f](g,h)$ is precisely the Hessian of f computed along the directions $\nabla g, \nabla h$.

Moreover by Theorem 1.4.8 we know that $\langle \nabla f, \nabla g\rangle \in \mathrm{D}(\mathbf{\Delta})$ for every $f, g \in \mathrm{Test}F(\mathrm{M})$ and so we can define the measure valued operator $\mathbf{\Gamma}_2\colon [\mathrm{Test}F(\mathrm{M})]^2 \to \mathrm{Meas}(\mathrm{M})$ as the map given by

$$\mathbf{\Gamma}_2(f,g) := \frac{1}{2}\mathbf{\Delta}\langle \nabla f, \nabla g\rangle - \frac{1}{2}\big(\langle \nabla f, \nabla \Delta g\rangle + \langle \nabla g, \nabla \Delta f\rangle\big)\mathfrak{m},$$

where as before we denote by $\mathrm{Meas}(\mathrm{M})$ the space of finite Borel measures on M equipped with the total variation norm. Again, in the smooth setting $\mathbf{\Gamma}_2(f,g)$ is always absolutely continuous with respect to the volume measure and its density is given by $\mathrm{Hess}(f) : \mathrm{Hess}(g) + \mathrm{Ric}(\nabla f, \nabla g)$. As for the non-smooth setting, $\mathbf{\Gamma}_2$ is bilinear and symmetric and the second inequality in (1.4.3.4) can be restated as

$$\mathbf{\Gamma}_2(f,f)(\mathrm{M}) \geq K|\nabla f|^2\mathfrak{m}. \tag{1.4.4.3}$$

Furthermore it holds

$$\mathbf{\Gamma}_2(f,f)(\mathrm{M}) = \int (\Delta f)^2 \,\mathrm{d}\mathfrak{m}, \qquad \|\mathbf{\Gamma}_2(f,f)\|_{\mathrm{TV}} \leq \int (\Delta f)^2 + 2K^-|\nabla f|^2 \,\mathrm{d}\mathfrak{m} \tag{1.4.4.4}$$

and the measure $\mathbf{\Gamma}_2(f,g)$ can be written as

$$\mathbf{\Gamma}_2(f,g) = \gamma_2(f,g)\mathfrak{m} + \mathbf{\Gamma}_2^s(f,g), \quad \text{where } \mathbf{\Gamma}_2^s(f,g)\perp\mathfrak{m}.$$

We then have the following crucial Lemma:

Lemma 1.4.12 (Key inequality)**.** *Let $n, m \in \mathbb{N}$ and $f_i, g_i, h_j \in \mathrm{Test}F(\mathrm{M})$, $i = 1, \dots, n, j = 1, \dots, m$. Define the measure $\mu = \mu((f_i),(g_i)) \in \mathrm{Meas}(\mathrm{M})$ by setting*

$$\begin{aligned}\mu((f_i),(g_i)) := \sum_{i,i'} &\bar g_i \bar g_{i'}\big(\mathbf{\Gamma}_2(f_i, f_{i'}) - K\langle \nabla f_i, \nabla f_{i'}\rangle\mathfrak{m}\big)\\ &+ \left(2g_i H[f_i](f_{i'}, g_{i'}) + \frac{\langle \nabla f_i, \nabla f_{i'}\rangle\langle \nabla g_i, \nabla g_{i'}\rangle + \langle \nabla f_i, \nabla g_{i'}\rangle\langle \nabla g_i, \nabla f_{i'}\rangle}{2}\right)\mathfrak{m},\end{aligned}$$

where $\bar g_i$ is the Lipschitz continuous representative of g_i, uniquely defined on $\mathrm{supp}(\mathfrak{m})$*, and write it as $\mu = \rho\mathfrak{m} + \mu^s$, where ρ is the density of μ with respect to $\mathfrak{m}$ and $\mu^s\perp\mathfrak{m}$ is the singular part of μ with respect to $\mathfrak{m}$.*

Then

$$\mu^s \geq 0 \tag{1.4.4.5}$$

and

$$\Big|\sum_{i,j}\langle \nabla f_i, \nabla h_j\rangle\langle \nabla g_i, \nabla h_j\rangle + g_i H[f_i](h_j, h_j)\Big|^2 \leq \rho\sum_{j,j'}|\langle \nabla h_j, \nabla h_{j'}\rangle|^2. \tag{1.4.4.6}$$

Actually (1.4.4.6) is nothing but (1.4.4.2) written for the test objects $X := \sum_i g_i \,\nabla f_i$ and $A := \sum_j \nabla h_j \otimes \nabla h_j$, $f_i, g_i, h_i \in \mathrm{Test}F(\mathrm{M})$.

The first important consequence of this lemma is given by the following Theorem, which guarantees that $\mathrm{Test}F(\mathrm{M}) \subset W^{2,2}(\mathrm{M})$ and that the space $W^{2,2}(\mathrm{M})$ is dense in $W^{1,2}(\mathrm{M})$.

Theorem 1.4.13. *Let $f \in \mathrm{Test}F(\mathrm{M})$. Then $f \in W^{2,2}(\mathrm{M})$ and*

$$|\mathrm{Hess}f|_{HS}^2 \leq \gamma_2(f,f) - K|\nabla f|^2, \quad \mathfrak{m}\text{-}a.e. \tag{1.4.4.7}$$

Moreover for every $g_1, g_2 \in \mathrm{Test}F(\mathrm{M})$ it holds

$$H[f](g_1,g_2) = \mathrm{Hess}f(\nabla g_1, \nabla g_2), \quad \mathfrak{m}\text{-}a.e. \tag{1.4.4.8}$$

In particular, from this theorem we deduce the following important corollary:

Corollary 1.4.14. *We have $\mathrm{D}(\Delta) \subset W^{2,2}(\mathrm{M})$ and*

$$\int |\mathrm{Hess}f|_{HS}^2 \,\mathrm{d}\mathfrak{m} \leq \int (\Delta f)^2 - K|\nabla f|^2 \,\mathrm{d}\mathfrak{m}, \quad \forall\ f \in \mathrm{D}(\Delta). \tag{1.4.4.9}$$

In turn, this result ensures that the following definition is meaningful:

Definition 1.4.15. *We define $H^{2,2}(\mathrm{M})$ as the $W^{2,2}$-closure of $\mathrm{D}(\Delta) \subset W^{2,2}(\mathrm{M})$.*

It is worth to underline that $H^{2,2}(\mathrm{M})$ also coincides with the $W^{2,2}$-closure of $\mathrm{Test}F(\mathrm{M})$ and we do not know whether $H^{2,2}(\mathrm{M})$ coincides with $W^{2,2}(\mathrm{M})$ or not.

Similarly we define $H^{2,2}_{\mathsf{loc}}(\mathrm{M})$ as the $W^{2,2}_{\mathsf{loc}}(\mathrm{M})$-closure of $\mathrm{Test}(\mathrm{M})$, i.e.: $f \in H^{2,2}_{\mathsf{loc}}(\mathrm{M}) \subset W^{2,2}_{\mathsf{loc}}(\mathrm{M})$ provided there exists a sequence $(f_n) \subset \mathrm{Test}(\mathrm{M})$ such that $f_n, \mathrm{d}f_n, \mathrm{Hess}(f_n)$ converge to $f, \mathrm{d}f, \mathrm{Hess}(f)$ in $L^2_{\mathsf{loc}}(\mathrm{M}), L^2_{\mathsf{loc}}(T^*\mathrm{M}), L^2_{\mathsf{loc}}((T^*)^{\otimes 2}\mathrm{M})$ respectively.

1.4.4.2 Calculus rules

We collect the basic calculus rules involving the Hessian and we refer to [36] for the proof of them:

- **Product rule for functions**: Let $f_1, f_2 \in \mathrm{Lip}_b \cap W^{2,2}(\mathrm{M})$, then $f_1 f_2 \in W^{2,2}(\mathrm{M})$ with

$$\mathrm{Hess}(f_1 f_2) = f_2 \mathrm{Hess}(f_1) + f_1 \mathrm{Hess}(f_2) + \mathrm{d}f_1 \otimes \mathrm{d}f_2 + \mathrm{d}f_2 \otimes \mathrm{d}f_1, \quad \mathfrak{m}\text{-a.e.} \tag{1.4.4.10}$$

- **Chain rule**: Let $f \in \mathrm{Lip} \cap W^{2,2}(\mathrm{M})$ and $\varphi\colon \mathbb{R} \to \mathbb{R}$ a C^2 function with uniformly bounded first and second derivative (and $\varphi(0) = 0$ in the case in which $\mathfrak{m}(\mathrm{M}) = +\infty$).

  Then $\varphi \circ f \in W^{2,2}(\mathrm{M})$ and the formula

$$\mathrm{Hess}(\varphi \circ f) = \varphi'' \circ f\ \mathrm{d}f \otimes \mathrm{d}f + \varphi' \circ f\ \mathrm{Hess}(f), \quad \mathfrak{m}\text{-a.e.} \tag{1.4.4.11}$$

  holds.

- **Product rule for gradients**: Let $f_1, f_2 \in \mathrm{Lip} \cap H^{2,2}(\mathrm{M})$. Then $\langle \nabla f_1, \nabla f_2 \rangle \in W^{1,2}(\mathrm{M})$ and

$$\mathrm{d}\langle \nabla f_1, \nabla f_2 \rangle = \mathrm{Hess}(f_1)(\nabla f_2, \cdot) + \mathrm{Hess}(f_2)(\nabla f_1, \cdot), \quad \mathfrak{m}\text{-a.e.} \tag{1.4.4.12}$$

- **Locality of the Hessian**: For every $f_1, f_2 \in W^{2,2}(\mathrm{M})$ we have

$$\mathrm{Hess}(f_1) = \mathrm{Hess}(f_2), \quad \mathfrak{m}\text{-a.e. on the interior of } \{f_1 = f_2\}, \tag{1.4.4.13}$$

  meaning the union of all the open sets $\Omega \subset \mathrm{M}$ such that $\{f_1 = f_2\}$ $\mathfrak{m}$-a.e. on Ω.

  Moreover for $f_1, f_2 \in H^{2,2}(\mathrm{M})$ the following finer property holds:

$$\mathrm{Hess}(f_1) = \mathrm{Hess}(f_2), \quad \mathfrak{m}\text{-a.e. on } \{f_1 = f_2\}. \tag{1.4.4.14}$$

- $H^{2,2}(\mathrm{M})$ **cut-off functions**: For every $B \subset \mathrm{M}$ and $\Omega \subset \mathrm{M}$ open with $\mathsf{d}(B, \Omega^c) > 0$ there exists $\eta_{B,\Omega} \in H^{2,2}(\mathrm{M})$ such that $\eta_{B,\Omega} = 1$ $\mathfrak{m}$-a.e. on B and $\eta_{B,\Omega} = 0$ $\mathfrak{m}$-a.e. on Ω^c.

  As for the proof of this approximation result we refer to [13] and to [43]. Briefly, in the first paper, the authors built for B and Ω as above a function in $\mathrm{Test}F(\mathrm{M})$ identically 1 on B and 0 on Ω^c and with bounded Laplacian, the argument being based on an approximation via heat flow. In the second paper instead the approach is different: in this case a condition on the dimension of the space is needed (it has to be finite) and the construction is based on the Laplacian comparison estimates for the distance (see [38]) and on the abstract Lewy-Stampacchia inequality.

1.4.5 Sobolev vector fields and covariant derivative

In this section we introduce the notion of Sobolev vector field, following the same approach used in the previous section for the definition of the Hessian.

Our starting point is the following identity, valid in a smooth Riemannian manifold for smooth functions g_1, g_2 and a smooth vector field X:

$$\langle \nabla_{\nabla g_1} X, \nabla g_2 \rangle = \langle \nabla(\langle X, \nabla g_2 \rangle), \nabla g_1 \rangle - \mathrm{Hess}(g_2)(X, \nabla g_1).$$

Actually in a smooth manifold, this equality together with the one defining the Hessian can be used as an alternative to Koszul formula in order to introduce the covariant derivative in terms of the metric tensor only, without the use of Lie brackets. This is convenient because there is no hope to define the Lie bracket for a generic vector field without imposing any sort of regularity to it, but to impose such regularity we need to already know in advance what the covariant derivative is.

Furthermore we point out that Sobolev regularity is the only kind of regularity we have for vector fields (namely we don't have any notion of Lipschitz or continuous vector field).

We recall that with $A : B$ we denote the pointwise scalar product of two tensors $A, B \in L^2(T^*\mathrm{M})$.

Definition 1.4.16 (The Sobolev space $W^{1,2}_C(T\mathrm{M})$ (resp. $W^{1,2}_{C,\mathsf{loc}}(T\mathrm{M})$))**.** *The Sobolev space $W^{1,2}_C(T\mathrm{M}) \subset L^2(T\mathrm{M})$ (resp. $W^{1,2}_{C,\mathsf{loc}}(T\mathrm{M}) \subset L^2_{\mathsf{loc}}(T\mathrm{M})$) is the space of all $X \in L^2(T\mathrm{M})$ (resp. $X \in L^2_{\mathsf{loc}}(T\mathrm{M})$) for which there exists $T \in L^2(T^{\otimes 2}\mathrm{M})$ (resp. $T \in L^2_{\mathsf{loc}}(T^{\otimes 2}\mathrm{M})$) such that for every $g_1, g_2 \in \mathrm{Test}F(\mathrm{M})$ (resp. $g_1, g_2 \in \mathrm{Test}F(\mathrm{M})$ with bounded support) and $h \in \mathrm{Lip}_b(\mathrm{M})$ it holds*

$$\int T : (\nabla g_1 \otimes \nabla g_2)\, \mathrm{d}\mathfrak{m} = \int -\langle X, \nabla g_2 \rangle \mathrm{div}(h \nabla g_1) - h \mathrm{Hess}(g_2)(X, \nabla g_1)\, \mathrm{d}\mathfrak{m}. \tag{1.4.5.1}$$

In this case we call the tensor T the covariant derivative of X and we denote it by ∇X. We endow $W^{1,2}_C(T\mathrm{M})$ with the norm $\|\cdot\|_{W^{1,2}_C(T\mathrm{M})}$ defined by

$$\|X\|^2_{W^{1,2}_C(T\mathrm{M})} := \|X\|^2_{L^2(T\mathrm{M})} + \|\nabla X\|^2_{L^2(T^{\otimes 2}\mathrm{M})}.$$

At this point it is useful to introduce the space of *test vector fields* as

$$\mathrm{Test}\, V(\mathrm{M}) := \left\{ \sum_{i=1}^n g_i \nabla f_i : n \in \mathbb{N}, f_i, g_i \in \mathrm{Test}F(\mathrm{M}) \right\} \subset L^2(T\mathrm{M}).$$

This set is dense in $L^2(T\mathrm{M})$ while the properties of test functions ensure that $\mathrm{Test}\, V(\mathrm{M}) \subset L^1 \cap L^\infty(T\mathrm{M})$ and that $\mathrm{Test}\, V(\mathrm{M}) \subset \mathrm{D}(\mathrm{div})$.

The basic properties of the space $W^{1,2}_C(T\mathrm{M})$ are collected in the following theorem, whose proof follows the proof of Theorem 1.4.10 and handles the calculus rules for the Hessian introduced in the previous section:

Theorem 1.4.17 (Basic properties of $W^{1,2}_C(T\mathrm{M})$)**.** *The following holds:*

i) $W^{1,2}_C(T\mathrm{M})$ *is a separable Hilbert space.*

ii) The covariant derivative is a closed operator, i.e., the set $\{(X,\nabla X) : X \in W^{1,2}_C(T\mathrm{M})\}$ *is a closed subset of* $L^2(T\mathrm{M}) \times L^2(T^{\otimes 2}\mathrm{M})$.

iii) Given $f \in W^{2,2}(\mathrm{M})$ *we have* $\nabla f \in W^{1,2}_C(\mathrm{M})$ *with* $\nabla(\nabla f) = \mathrm{Hess}(f)^\sharp$.

iv) We have $\mathrm{Test}\,V(\mathrm{M}) \subset W^{1,2}_{\mathrm{C}}(T\mathrm{M})$ *with*

$$\nabla X = \sum_i \nabla g_i \otimes \nabla f_i + g_i(\mathrm{Hess} f_i)^\sharp, \quad \textit{for } X = \sum_i g_i \nabla f_i,$$

and this implies that $W^{1,2}_C(T\mathrm{M})$ *is dense in* $L^2(T\mathrm{M})$.

1.4.5.1 Calculus rules

By Theorem 1.4.17, the space $\mathrm{Test}\,V(\mathrm{M})$ is contained in $W^{1,2}_C(T\mathrm{M})$, but we don't know if it is dense. Then the following definition is meaningful:

Definition 1.4.18. *We define* $H^{1,2}_C(T\mathrm{M}) \subset W^{1,2}_C(T\mathrm{M})$ *as the* $W^{1,2}_C(T\mathrm{M})$*-closure of* $\mathrm{Test}\,V(\mathrm{M})$.

The space $H^{1,2}_{C,\mathsf{loc}}(T\mathrm{M})$ is then equivalently defined either as the subspace of $L^2_{\mathsf{loc}}(T\mathrm{M})$ made of vectors X of such that $fX \in H^{1,2}_C(T\mathrm{M})$ for every $f \in \mathrm{Test}(\mathrm{M})$ with bounded support or as the $W^{1,2}_{C,\mathsf{loc}}$-closure of $H^{1,2}_C(T\mathrm{M})$, i.e. as the space of vector fields $X \in W^{1,2}_{C,\mathsf{loc}}(T\mathrm{M})$ such that there is $(X_n) \subset H^{1,2}_C(T\mathrm{M})$ such that $X_n \to X$ and $\nabla X_n \to \nabla X$ in $L^2_{\mathsf{loc}}(T\mathrm{M})$ and $L^2_{\mathsf{loc}}(T^{\otimes 2}\mathrm{M})$ as $n \to \infty$.

In the following we shall denote by $L^0(T\mathrm{M})$ the L^0-completion of $L^2(T\mathrm{M})$ (see Theorem 1.2.5) and by $L^\infty(T\mathrm{M})$ its subspace made of vector fields X with $|X| \in L^\infty(\mathfrak{m})$.

A first useful result is the following:

Proposition 1.4.19 (Leibniz rule)**.** *Let* $X \in L^\infty \cap W^{1,2}_C(T\mathrm{M})$ *and* $f \in L^\infty \cap W^{1,2}(\mathrm{M})$. *Then* $fX \in W^{1,2}_C(T\mathrm{M})$ *and*

$$\nabla(fX) = \nabla f \otimes X + f\nabla X, \quad \mathfrak{m}\text{-}a.e. \tag{1.4.5.2}$$

Then we introduce the following notation: for $X \in W^{1,2}_C(T\mathrm{M})$ and $Z \in L^\infty(T\mathrm{M})$, the vector field $\nabla_Z X \in L^2(T\mathrm{M})$ is defined by

$$\langle \nabla_Z X, Y\rangle := \nabla X : (X \otimes Y), \quad \mathfrak{m}\text{-a.e.}, \quad \forall\, Y \in L^2(T\mathrm{M}),$$

where the right hand side is firstly defined for $Z, Y \in L^0(L^2(T\mathrm{M}))$ such that $Z \otimes Y \in L^2(T^{\otimes 2}\mathrm{M})$ and then extended by continuity to a bilinear map from $[L^0(T\mathrm{M})]^2$ to $L^0(\mathfrak{m})$.

In the next two propositions we see that, in an appropriate sense, the covariant derivative satisfies the axioms of the Levi-Civita connection.

Proposition 1.4.20 (Compatibility with the metric)**.** *Let* $X, Y \in L^\infty \cap H^{1,2}_C(T\mathrm{M})$. *Then* $\langle X, Y\rangle \in W^{1,2}(\mathrm{M})$ *and*

$$\mathrm{d}\langle X,Y\rangle(Z) = \langle \nabla_Z X, Y\rangle + \langle \nabla_Z Y, X\rangle, \quad \mathfrak{m}\text{-}a.e.$$

for every $Z \in L^2(T\mathrm{M})$.

We now pass to the torsion-free identity, which, as in the smooth setting, follows directly from the symmetry of the Hessian and the compatibility of the metric. Indeed observe that from the definition of $H^{1,2}_C(T\mathrm{M})$, we have $\nabla f \in L^\infty \cap H^{1,2}_C(T\mathrm{M})$, while Proposition 1.4.20 ensures that $Y(f) \in W^{1,2}(\mathrm{M})$. Then a direct computation shows that:

Proposition 1.4.21 (Torsion free identity). *Let $f \in \mathrm{Lip} \cap H^{2,2}(\mathrm{M})$ and $X, Y \in L^\infty \cap H^{1,2}_C(T\mathrm{M})$. Then $X(f)$ and $Y(f)$ are in $W^{1,2}(\mathrm{M})$ and*

$$X(Y(f)) - Y(X(f)) = \mathrm{d}f(\nabla_X Y - \nabla_Y X), \quad \mathfrak{m}\text{-a.e.} \tag{1.4.5.3}$$

where we have used the notation $X(f)$ in place of $\mathrm{d}f(X)$.

Since Test$F(\mathrm{M})$ is dense in $W^{1,2}(\mathrm{M})$, the same holds for $H^{2,2}(\mathrm{M})$ and recalling that the cotangent module $L^2(T^*\mathrm{M})$ is generated, in the sense of modules, by the space $\{\mathrm{d}f : f \in W^{1,2}(\mathrm{M})\}$ (point (*ii*) in Theorem 1.3.3), we see that $L^2(T^*\mathrm{M})$ is also generated by $\{\mathrm{d}f : f \in H^{2,2}(\mathrm{M})\} \subset L^\infty(T^*\mathrm{M})$. Hence the vector field $\nabla_X Y - \nabla_Y X$ is the only one for which the identity (1.4.5.3) holds for any $f \in H^{2,2}(\mathrm{M})$ and it is meaningful to define:

Definition 1.4.22 (Lie bracket of Sobolev vector fields). *For every couple of vector fields $X, Y \in W^{1,2}_C(T\mathrm{M})$ we define the Lie bracket as*

$$[X, Y] := \nabla_X Y - \nabla_Y X \in L^1(T\mathrm{M}) \tag{1.4.5.4}$$

We conclude this section discussing the locality properties of the covariant derivative. As we have done for the locality of the Hessian, also in this case we need the notion of the 'interior of $\{X_1 = X_2\}$ for vector fields X_1, X_2 which again is nothing but the union of all the open sets $\Omega \subset \mathrm{M}$ such that $X_1 = X_2$ $\mathfrak{m}$-a.e. on Ω.

Proposition 1.4.23 (Locality of the covariant derivative). *For any $X_1, X_2 \in W^{1,2}_C(T\mathrm{M})$ we have*

$$\nabla X_1 = \nabla X_2, \quad \mathfrak{m}\text{-a.e. on the interior of } \{X_1 = X_2\}, \tag{1.4.5.5}$$

while if $X_1, X_2 \in H^{1,2}_C(T\mathrm{M})$ the following stronger property holds

$$\nabla X_1 = \nabla X_2, \quad \mathfrak{m}\text{-a.e. on } \{X_1 = X_2\}. \tag{1.4.5.6}$$

1.4.6 Sobolev differential forms and calculus rules

In this section we use the construction done for the exterior power of a Hilbert module (Section 1.2.6) in the case in which $\mathscr{H} = L^2(T^*\mathrm{M})$: we write $L^2(\Lambda^k T^*\mathrm{M})$ for the k-th exterior power if $k > 1$, while we write $L^2(T^*\mathrm{M})$ if $k = 1$and $L^2(\mathfrak{m})$ if $k = 0$. We shall refer to elements in $L^2(\Lambda^k T^*\mathrm{M})$ as k-forms.

Moreover the duality between $L^2(T^*\mathrm{M})$ and $L^2(T\mathrm{M})$ induces a duality relation between the respective k-th exterior powers; we will write $\omega(X_1, \dots, X_k)$ in place of $\omega(X_1 \wedge \dots \wedge X_k)$.

As before, in order to define the exterior differential of a k-form ω we start our study from the smooth setting, where it is characterized by the identity

$$\begin{aligned} \mathrm{d}\omega(X_0, \dots, X_k) = & \sum_i (-1)^i \ \mathrm{d}(\omega(X_0, \dots, \hat{X}_i, \dots, X_k))(X_i) \\ & + \sum_{i<j} (-1)^{i+j} \ \omega([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k), \end{aligned}$$

for any smooth vector fields $X_1, \ldots, X_k$.

Since in the previous section we have defined the Lie bracket for a large class of vector fields, in which the test vector fields $\mathrm{Test}\, V(\mathrm{M})$ are included, we can use the above formula in order to define the exterior differential of forms.

Notice that for $X_i \in \mathrm{Test}\, V(\mathrm{M})$ the fact that $X_i \in L^2 \cap L^\infty(T\mathrm{M})$ guarantees that $|X_1 \wedge \cdots \wedge X_n| \in L^2(\mathfrak{m})$ as well as $|[X_i, X_j], X_1 \wedge \cdots \wedge X_n| \in L^2(\mathfrak{m})$. Moreover this ensures that

$$\text{the linear span of } \{X_1 \wedge \cdots \wedge X_n : X_1, \ldots, X_n \in \mathrm{Test}\, V(\mathrm{M})\} \text{ is dense in } L^2(\Lambda^k T^*\mathrm{M}). \tag{1.4.6.1}$$

Then we give the following definition:

Definition 1.4.24 (The space $W^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{M})$)**.** *The space $W^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{M}) \subset L^2(\Lambda^k T^*\mathrm{M})$ is the space of k-forms ω such that there exists a $k+1$ form $\eta \in L^2(\Lambda^{k+1} T^*\mathrm{M})$ for which the identity*

$$\begin{aligned}\int \eta(X_0, \ldots, X_k)\, \mathrm{d}\mathfrak{m} = &\int \sum_i (-1)^{i+1}\ \omega(X_0, \ldots, \hat{X}_i, \ldots, X_k) \mathrm{div}(X_i)\, \mathrm{d}\mathfrak{m} \\ &+ \int \sum_{i<j} (-1)^{i+j}\ \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k)\, \mathrm{d}\mathfrak{m},\end{aligned} \tag{1.4.6.2}$$

holds for any $X_0, \ldots, X_k \in \mathrm{Test}\, V(\mathrm{M})$. In this case we call η the exterior differential of ω and we denote it by $\mathrm{d}\omega$.

We endow $W^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{M})$ with the norm $\|\cdot\|_{W^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{M})}$ given by

$$\|\omega\|^2_{W^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{M})} := \|\omega\|^2_{L^2(\Lambda^k T^*\mathrm{M})} + \|\mathrm{d}\omega\|^2_{L^2(\Lambda^{k+1} T^*\mathrm{M})}.$$

We remark that the density property in (1.4.6.1) grants that for $\omega \in W^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{M})$ the exterior differential $\mathrm{d}\omega$ is unique and linearly depends on ω, so that $W^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{M})$ is a normed vector space.

We also point out that $W^{1,2}_{\mathrm{d}}(\Lambda^0 T^*\mathrm{M}) = W^{1,2}(\mathrm{M})$ and for a function in this space the definitions of differential as given above and in Definition 1.3.3 coincide, namely a function $f \in L^2(\mathfrak{m})$ is actually in $W^{1,2}(\mathrm{M})$ if and only if there exists $\omega \in L^2(T^*\mathrm{M})$ such that $\int f \mathrm{div}(X)\, \mathrm{d}\mathfrak{m} = -\int \omega(X)\, \mathrm{d}\mathfrak{m}$ for any $X \in \mathrm{Test}\, V(\mathrm{M})$ and in this case ω is the differential of f as introduced in Definition 1.3.3.

We then have the following:

Theorem 1.4.25 (Basic properties of $W^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{M})$)**.** *For every $k \in \mathbb{N}$ the following holds:*

i) $W^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{M})$ is a separable Hilbert space.

ii) The exterior differential is a closed operator, i.e., $\{(\omega, \mathrm{d}\omega) : \omega \in W^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{M})\}$ is a closed subspace of $L^2(\Lambda^k T^*\mathrm{M}) \times L^2(\Lambda^{k+1} T^*\mathrm{M})$.

Also in the case of Sobolev forms, it is convenient to introduce the space of *test k-forms* as

$$\begin{aligned}\mathrm{TestForm_k}(\mathrm{M}) := \{&\text{linear combinations of formsof the type } f_0 \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k, \\ &\text{with } f_i \in \mathrm{Test}F(\mathrm{M})\ \forall\, i = 0, \ldots, k\}.\end{aligned}$$

Using the fact that $\mathrm{Test}\, V(\mathrm{M})$ is dense in $L^2(T^{\otimes 2}\mathrm{M})$ and recalling the fact that $\mathrm{Test}F(\mathrm{M})$ is an algebra and that the exterior product is obtained as a quotient of the tensor product, we can prove that $\mathrm{TestForm_k}(\mathrm{M})$ is dense in $L^2(\Lambda^k T^*\mathrm{M})$ for every $k \in \mathbb{N}$.

Then the following result is useful:

Proposition 1.4.26 (Basic calculus rules for exterior differentiation). *The following holds:*

i) For $f_i \in L^\infty \cap W^{1,2}(\mathrm{M})$ with $|\mathrm{d}f_i| \in L^\infty$, $i = 0, \dots, k$ we have that $f_0 \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k \in W^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{M})$ and

$$\mathrm{d}(f_0 \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k) = \mathrm{d}f_0 \wedge \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k, \tag{1.4.6.3}$$

and similarly $\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k \in W^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{M})$ with

$$\mathrm{d}(\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k) = 0. \tag{1.4.6.4}$$

ii) We have that $\mathrm{TestForm_k}(\mathrm{M}) \subset W^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{M})$ and in particular $W^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{M})$ is dense in $L^2(\Lambda^k T^*\mathrm{M})$.

iii) Let $\omega \in W^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{M})$ and $\omega' \in \mathrm{TestForm_k}(\mathrm{M})$. Then $\omega \wedge \omega' \in L^2(\Lambda^{k+k'} T^*\mathrm{M})$ with

$$\mathrm{d}(\omega \wedge \omega') = \mathrm{d}\omega \wedge \omega' + (-1)^k \omega \wedge \mathrm{d}\omega'.$$

iv) (Leibniz rule) *Let $\omega \in W^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{M})$ and $\omega' \in \mathrm{TestForm_{k'}}(\mathrm{M})$. Then $\omega \wedge \omega' \in W^{1,2}_{\mathrm{d}}(\Lambda^{k+k'} T^*\mathrm{M})$ with*

$$\mathrm{d}(\omega \wedge \omega') = \mathrm{d}\omega \wedge \omega' + (-1)^k \omega \wedge \mathrm{d}\omega'. \tag{1.4.6.5}$$

Remark 1.4.27. It is worth to point out that on an arbitrary infinitesimally Hilbertian space we could have defined the space $L^2(\Lambda^k T^*\mathrm{M})$ as before and declared that for Sobolev functions $f_0, \dots, f_k$ with appropriate integrability the exterior differential of the form $f_0 \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k$ is given by $\mathrm{d}f_0 \wedge \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k$.

However without further assumptions it is not clear whether the so defined differential is closable. Instead in the approach explained above the closure of the exterior differential is granted by the assumption on the lower bound on the Ricci curvature: indeed this hypothesis guarantees the existence of a large class of vector fields for which the Lie bracket is well defined and this in turn allows to give the definition of exterior differential via integration by parts, which directly leads to the expected closure of the differential.

Thanks to Proposition 1.4.26 the following definition is meaningful:

Definition 1.4.28 (The space $H^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{M})$). *We define $H^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{M}) \subset W^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{M})$ as the $W^{1,2}_{\mathrm{d}}$-closure of $\mathrm{TestForm_k}(\mathrm{M})$.*

We remark that point (*ii*) in Proposition 1.4.26 ensures that $H^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{M})$ is dense in $L^2(\Lambda^k T^*\mathrm{M})$. A crucial property of forms in $H^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{M})$ is:

Proposition 1.4.29 ($\mathrm{d}^2 = 0$ for forms in $H^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{M})$). *Let $\omega \in H^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{M})$. Then*

$$\mathrm{d}\omega \in H^{1,2}_{\mathrm{d}}(\Lambda^{k+1} T^*\mathrm{M}) \quad \textit{and} \quad \mathrm{d}(\mathrm{d}\omega) = 0.$$

Finally we specify that the exterior differential is local in the following sense:

$$\forall \omega \in W^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{M}) \text{ we have } \mathrm{d}\omega = 0 \;\; \mathfrak{m}\text{-a.e. on the interior of } \{\omega = 0\}, \tag{1.4.6.6}$$

where the 'interior of $\{\omega = 0\}$' is by definition the union of all the open sets $\Omega \subset \mathrm{M}$ such that $\omega = 0$ $\mathfrak{m}$-a.e. on Ω. However by now we don't know if it is possible to improve (1.4.6.6) for $\omega \in H^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{M})$ into 'dω is 0 where ω is zero' as we did for the Hessian and the covariant derivative. The technical problem is the fact that for $\omega \in H^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{M})$ and $X_i \in \mathrm{Test}\, V(\mathrm{M})$ it is not clear whether the function $\omega(X_1, \dots, X_k)$ has any kind of Sobolev regularity.

1.4.7 de Rham cohomology and Hodge theory

In the previous section we have defined the exterior differential and proved that, at least for forms in $H^{1,2}_{\rm d}(\Lambda^k T^*{\rm M})$, it squares to 0. In fact Proposition 1.4.29 is our starting point for building the de Rham complex.

First of all we define the spaces $C_k({\rm M})$ and ${\rm E}_k({\rm M})$ of closed and exact k-forms as:

$$
\begin{aligned}
{\rm C}_k({\rm M}) &:= \ker\Big({\rm d}\colon H^{1,2}_{\rm d}(\Lambda^k T^*{\rm M}) \to H^{1,2}_{\rm d}(\Lambda^{k+1} T^*{\rm M})\Big) = \{\omega \in H^{1,2}_{\rm d}(\Lambda^k T^*{\rm M}) : {\rm d}\omega = 0\},\\
{\rm E}_k({\rm M}) &:= {\rm Im}\Big({\rm d}\colon H^{1,2}_{\rm d}(\Lambda^{k-1} T^*{\rm M}) \to H^{1,2}_{\rm d}(\Lambda^{k} T^*{\rm M})\Big) = \{{\rm d}\omega : \omega \in H^{1,2}_{\rm d}(\Lambda^{k-1} T^*{\rm M})\}.
\end{aligned}
$$

The closure of the differential guarantees that ${\rm C}_k({\rm M})$ is a closed subspace of $L^2(\Lambda^k T^*{\rm M})$, while we don't know whether ${\rm E}_k({\rm M})$ is closed. Thus we introduce the space $\overline{\rm E}_k({\rm M})$ as

$$\overline{\rm E}_k({\rm M}) := L^2(\Lambda^k T^*{\rm M})\text{-closure of } {\rm E}_k({\rm M}).$$

Since Proposition 1.4.29 ensures that ${\rm E}_k({\rm M}) \subset {\rm C}_k({\rm M})$, we also have $\overline{\rm E}_k({\rm M}) \subset {\rm C}_k({\rm M})$. Hence we define:

Definition 1.4.30 (se Rham cohomology)**.** *For $k \in \mathbb{N}$ the vector space $\mathcal{H}^k_{dR}({\rm M})$ is defined as the quotient*

$$\mathcal{H}^k_{dR} := \frac{{\rm C}_k({\rm M})}{\overline{\rm E}_k({\rm M})}.$$

Endowing ${\rm C}_k({\rm M})$ and $\overline{\rm E}_k({\rm M})$ with the $L^2(\Lambda^k T^*{\rm M})$-norm, they become both Hilbert spaces and then $\mathcal{H}^k_{\rm dR}$ comes with a canonical structure of Hilbert space as well. Actually in our setup this Hilbert structure is intrinsic, being based on the existence of the pointwise scalar product defined in the tangent module of our base space $({\rm M}, \mathsf{d}, \mathfrak{m})$, which allows to define the concept of k-forms and of their differentials.

We turn then to functoriality; let $\varphi\colon {\rm M}_2 \to {\rm M}_1$ be a map of bounded deformation and recall that in Definition/Theorem 1.3.20 we have introduced the definition of pullback of 1-forms $\varphi^*\colon L^2(T^*{\rm M}_1) \to L^2(T^*{\rm M}_2)$, which is a map characterized by

$$\varphi^*({\rm d}f) = {\rm d}(f \circ \varphi), \tag{1.4.7.1a}$$
$$\varphi^*(g\omega) = g \circ \varphi\ \varphi^*\omega \tag{1.4.7.1b}$$
$$|\varphi^*\omega| \le {\rm Lip}(\varphi)|\omega| \circ \varphi \tag{1.4.7.1c}$$

valid $\mathfrak{m}_2$-a.e., for every $f \in \mathcal{S}^2({\rm M}_1)$, $\omega \in L^2(T^*{\rm M}_1)$ and $g \in L^\infty(\mathfrak{m}_1)$.

The construction we are going to present below is devoted to extending the pullback operation to general forms in $L^2(\Lambda^k T^*{\rm M}_1)$ for $k \in \mathbb{N}$.

First of all we observe that thanks to (1.4.7.1c) we can uniquely extend φ^* to a linear continuous map from $L^0(T^*{\rm M}_1)$ to $L^0(T^*{\rm M}_2)$ still satisfying (1.4.7.1b) and (1.4.7.1c) for arbitrary $g \in L^0(\mathfrak{m}_1)$ and $\omega \in L^0(T^*{\rm M}_1)$. Then we claim that for every $\omega_1, \dots, \omega_k \in L^2(T^*{\rm M}_1)$ we have

$$|(\varphi^*\omega_1) \wedge \cdots \wedge (\varphi^*\omega_k)| \le {\rm Lip}(\varphi)^k |\omega_1 \wedge \cdots \wedge \omega_k| \circ \varphi, \qquad \mathfrak{m}_2\text{−}a.e. \tag{1.4.7.2}$$

In order to prove this inequality we recall the structural characterization of Hilbert modules given in Theorem 1.2.19 to reduce the study to Hilbert spaces. Moreover, since (1.4.7.2) involves only a finite number of vectors we can also reduce to finite dimensional Hilbert spaces. Therefore (1.4.7.2) is proved once we have shown that if H is an n-dimensional Hilbert space and A

is an endomorphism of H, then the induced endomorphism $A^{\wedge^k}\colon \Lambda^k H \to \Lambda^k H$, defined by $A^{\wedge^k}(v_1 \wedge \cdots \wedge v_k) := Av_1 \wedge \cdots \wedge Av_k$ and then extended by linearity, has operator norm bounded by the k-th power of the operator norm of A. This can be proved introducing the symmetric and positively defined operator $B := A^t A$: let $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ be its eigenvalues and recall that the operator norm of A is equal to $\sqrt{\lambda_1}$. We denote by $v_1, \dots, v_n \in H$ the set of orthogonal eigenvectors of B and we notice that for any choice of distinct $i_1, \dots, i_k$ we have that $v_{i_1} \wedge \ldots \wedge v_{i_k} \in \Lambda^k H$ is an eigenvector of $(A^{\wedge^k})^t A^{\wedge^k} = B^{\wedge^k}$, the corresponding eigenvalue being $\prod_j \lambda_{i_j}$. Since these are $\binom{n}{k}$ independent elements of $\Lambda^k H$, in such a way we have found all the eigenvectors of $B^{\wedge^k}$ and seen that the norm of each of them is bounded by λ_1^k, which proves the claim.

The bound in (1.4.7.2) guarantees the existence of a unique linear map $\varphi^*\colon L^0(\Lambda^k T^*\mathrm{M}_1) \to L^0(\Lambda^k T^*\mathrm{M}_2)$ such that

$$\begin{gathered}\varphi^*(\omega_1 \wedge \cdots \wedge \omega_k) = \varphi^*(\omega_1) \wedge \cdots \wedge \varphi^*(\omega_k) \\ |\varphi^*\omega| \leq \mathrm{Lip}(\varphi)^k |\omega| \circ \varphi,\end{gathered} \tag{1.4.7.3}$$

$\mathfrak{m}_2$-a.e. for every $\omega_1, \dots, \omega_k \in L^0(T^*\mathrm{M}_1)$ and $\omega \in L^0(\Lambda^k T^*\mathrm{M}_1)$. Hence we can restrict φ^* to a map still denoted by φ^* from $L^2(\Lambda^k T^*\mathrm{M}_1)$ to $L^2(\Lambda^k T^*\mathrm{M}_2)$ still satisfying the properties in (1.4.7.3) plus

$$\varphi^*(f\omega) = f \circ \varphi\, \varphi^*\omega \tag{1.4.7.4}$$

for every $\omega \in L^2(\Lambda^k T^*\mathrm{M}_1)$ and $f \in L^\infty(\mathfrak{m}_1)$.

Proposition 1.4.31 (Functoriality)**.** *Let $(\mathrm{M}_1, \mathsf{d}_1, \mathfrak{m}_1)$ and $(\mathrm{M}_2, \mathsf{d}_2, \mathfrak{m}_2)$ be two $\mathsf{RCD}(K,\infty)$ spaces, $K \in \mathbb{R}$, and $\varphi\colon \mathrm{M}_2 \to \mathrm{M}_1$ of bounded deformation. Then for every $k \in \mathbb{N}$ and $\omega \in H^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{M}_1)$ we have $\varphi^*\omega \in H^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{M}_2)$ with*

$$\mathrm{d}(\varphi^*\omega) = \varphi^*\mathrm{d}\omega. \tag{1.4.7.5}$$

In particular, φ^* passes to the quotient and induces a linear continuous map from $\mathcal{H}^k_{dR}(\mathrm{M}_1)$ to $\mathcal{H}^k_{dR}(\mathrm{M}_2)$ with norm bounded by $\mathrm{Lip}(\varphi)^k$.

proof The linearity and continuity of φ^* and of $\mathrm{d}\colon H^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{M}_2) \to H^{1,2}_{\mathrm{d}}(\Lambda^{k+1} T^*\mathrm{M}_2)$ ensures that it is sufficient to prove (1.4.7.5) for ω of the form $\omega = f_0\, \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k$, for $f_i \in \mathrm{Test}F(\mathrm{M}_1)$. In this case the definition of φ^* together with equality in (1.4.7.1a) guarantee that

$$\varphi^*\omega = f_0 \circ \varphi \mathrm{d}(f_1 \circ \varphi) \wedge \cdots \wedge \mathrm{d}(f_k \circ \varphi)$$

and since $f_i \circ \varphi \in L^\infty \cap W^{1,2}(\mathrm{M}_2)$ with $|\mathrm{d}(f_i \circ \varphi)| \in L^\infty(\mathfrak{m}_2)$, from point (i) in Proposition 1.4.26 we deduce that

$$\mathrm{d}\varphi^*\omega = \mathrm{d}(f_0 \circ \varphi) \wedge \mathrm{d}(f_1 \circ \varphi) \wedge \cdots \wedge \mathrm{d}(f_k \circ \varphi) = \varphi^*\mathrm{d}\omega,$$

which is the claim.

The only thing left to prove is the fact that φ^* induces a linear continuous map from $\mathcal{H}^k_{\mathrm{dR}}(\mathrm{M}_1)$ to $\mathcal{H}^k_{\mathrm{dR}}(\mathrm{M}_2)$ with norm bounded by $\mathrm{Lip}(\varphi)^k$. For that purpose notice that (1.4.7.5) ensures that φ^* sends closed (resp. exact) forms in closed (resp. exact) forms and that its continuity guarantees that forms in $\overline{\mathrm{E}}_k(\mathrm{M}_1)$ are sent in forms in $\overline{\mathrm{E}}_k(\mathrm{M}_2)$. This means that φ^* passes to the quotient. Finally, the bound on the norm is then a direct consequence of the second inequality (1.4.7.3). $\square$

We now turn to *Hodge theory* about representation of cohomology classes via harmonic forms. We observe that the theory valid for compact smooth manifolds actually fits in our framework, being it based on the notion of L^2 and Sobolev forms. In order to introduce it, we shall need a few definitions:

Definition 1.4.32 (Codifferential). *The space* $\mathrm{D}(\delta_k) \subset L^2(\Lambda^k T^*\mathrm{M})$ *is the space of those forms* ω *for which there exists a form* $\delta\omega \in L^2(\Lambda^{k-1} T^*\mathrm{M})$, *called codifferential of* ω, *such that*

$$\int \langle \delta\omega, \eta\rangle \, \mathrm{d}\mathfrak{m} = \int \langle \omega, \mathrm{d}\eta\rangle \, \mathrm{d}\mathfrak{m}, \qquad \forall\ \eta \in \mathrm{TestForm_{k-1}(M)}. \tag{1.4.7.6}$$

In the case in which $k = 0$ *we put* $\mathrm{D}(\delta_0) := L^2(\mathfrak{m})$ *and we define the operator* δ *to be identically 0 on it.*

The density of $\mathrm{TestForm_{k-1}(M)}$ in $L^2(\Lambda^{k-1} T^*\mathrm{M})$ ensures that the codifferential is uniquely defined, while directly from the definition we deduce that δ is a closed operator in the sense that $\{(\omega, \delta\omega) : \omega \in \mathrm{D}(\delta_k)\}$ is a closed subspace of $L^2(\Lambda^k T^*\mathrm{M}) \times L^2(\Lambda^{k-1} T^*\mathrm{M})$.

In the case of 1-forms, the operator δ is nothing but the opposite of the divergence, meaning that $\omega \in \mathrm{D}(\delta_1)$ if and only if $\omega^\sharp \in \mathrm{D}(\mathrm{div})$ and in this case

$$\delta\omega = -\mathrm{div}(\omega^\sharp). \tag{1.4.7.7}$$

Moreover we remark that the codifferential is a local operator, i.e., for any $\omega \in \mathrm{D}(\delta_k)$ it holds

$$\delta\omega = 0 \quad \mathfrak{m}\text{-a.e. on the interior of } \{\omega = 0\}, \tag{1.4.7.8}$$

meaning that '$\delta\omega = 0$ $\mathfrak{m}$-a.e. on every open set Ω on which ω is $\mathfrak{m}$-a.e. 0'.

Some technical computations show that actually $\mathrm{TestForm_k(M)}$ is a subset of $\mathrm{D}(\delta)$, while we recall that from point (*ii*) in Theorem 1.4.26, $\mathrm{TestForm_k(M)} \subset W^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{M})$. Therefore we can introduce the 'Hodge' Sobolev space:

Definition 1.4.33 (The space $W^{1,2}_H(\Lambda^k T^*\mathrm{M})$ and $H^{1,2}_H(\Lambda^k T^*\mathrm{M})$). *For* $k \in \mathbb{N}$ *we define the space* $W^{1,2}_H(\Lambda^k T^*\mathrm{M}) := W^{1,2}_{\mathrm{d}}(\Lambda^k T^*\mathrm{M}) \cap \mathrm{D}(\delta)$ *endowed with the norm*

$$\|\omega\|^2_{W^{1,2}_H(\Lambda^k T^*\mathrm{M})} := \|\omega\|^2_{L^2(\Lambda^k T^*\mathrm{M})} + \|\mathrm{d}\omega\|^2_{L^2(\Lambda^{k+1} T^*\mathrm{M})} + \|\delta\omega\|_{L^2(\Lambda^{k-1} T^*\mathrm{M})},$$

and the space $H^{1,2}_H(\Lambda^k T^*\mathrm{M})$ *as the* $W^{1,2}_H$*-closure of* $\mathrm{TestForm_k(M)}$.

The Hodge energy functional $\mathsf{E}_H\colon L^2(\Lambda^k T^*\mathrm{M}) \to [0,\infty]$ *is defined by*

$$\mathsf{E}_H(\omega) := \begin{cases} \dfrac{1}{2}\displaystyle\int |\mathrm{d}\omega|^2 + |\delta\omega|^2 \, \mathrm{d}\mathfrak{m}, & \text{if } \omega \in W^{1,2}_H(\Lambda^k T^*\mathrm{M}), \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.4.7.9}$$

Also in this case it is possible to prove that $W^{1,2}_H(\Lambda^k T^*\mathrm{M})$ and $H^{1,2}_H(\Lambda^k T^*\mathrm{M})$ are separable Hilbert spaces and E_H is lower semicontinuous. In particular $H^{1,2}_H(\Lambda^k T^*\mathrm{M})$ is dense in $L^2(\Lambda^k T^*\mathrm{M})$.

Definition 1.4.34 (Hodge Laplacian and harmonic forms). *Given* $k \in \mathbb{N}$, *the domain* $\mathrm{D}(\Delta_{H,k}) \subset H^{1,2}_H(\Lambda^k T^*\mathrm{M})$ *of the Hodge Laplacian is the set of* $\omega \in H^{1,2}_H(\Lambda^k T^*\mathrm{M})$ *for which there exists* $\alpha \in L^2(\Lambda^k T^*\mathrm{M})$ *such that*

$$\int \langle \alpha, \eta\rangle \, \mathrm{d}\mathfrak{m} = \int \langle \mathrm{d}\omega, \mathrm{d}\eta\rangle + \langle \delta\omega, \delta\eta\rangle \, \mathrm{d}\mathfrak{m}, \qquad \forall\ \eta \in H^{1,2}_H(\Lambda^k T^*\mathrm{M}).$$

In this case the form α *(which is unique by the density of* $H^{1,2}_H(\Lambda^k T^*\mathrm{M})$ *in* $L^2(\Lambda^k T^*\mathrm{M})$*) will be called Hodge Laplacian of* ω *and denoted by* $\Delta_H\omega$.

The space $Harm_k(\mathrm{M}) \subset \mathrm{D}(\Delta_{H,k})$ *is the space of forms* $\omega \in \mathrm{D}(\Delta_{H,k})$ *such that* $\Delta_H\omega = 0$.

We remark that for every $f \in L^2(\mathfrak{m})$ we have $\delta f = 0$, which in particular means that $\mathrm{D}(\Delta_{H,0}) = \mathrm{D}(\Delta)$ with

$$\Delta_H f = -\Delta f \qquad \forall\, f \in \mathrm{D}(\Delta) \subset L^2(\Lambda^0 T^*\mathrm{M}) = L^2(\mathfrak{m}).$$

Directly from the definition of Hodge energy functional we see that

$$\mathsf{E}_H(\omega) = \frac{1}{2}\int \langle \omega, \Delta_H\omega\rangle\,\mathrm{d}\mathfrak{m}, \quad \forall\, \omega \in \mathrm{D}(\Delta_{H,k}). \tag{1.4.7.10}$$

In particular the Hodge Laplacian can also be seen as the only element in the subdifferential of the augmented Hodge energy $\tilde{\mathsf{E}}_H \colon L^2(\Lambda^k T^*\mathrm{M}) \to [0,\infty]$, which is a convex and lower semicontinuous functional defined by

$$\tilde{\mathsf{E}}_H(\omega) := \begin{cases} \dfrac{1}{2}\displaystyle\int |\mathrm{d}\omega|^2 + |\delta\omega|^2\,\mathrm{d}\mathfrak{m}, & \text{if } \omega \in H^{1,2}_H(\Lambda^k T^*\mathrm{M}),\\ +\infty, & \text{otherwise.} \end{cases} \tag{1.4.7.11}$$

From this equivalent characterization we see that Δ_H is a closed operator, i.e., $\{(\omega,\Delta_H\omega) : \omega \in \mathrm{D}(\Delta_{H,k})\}$ is a closed subset of $L^2(\Lambda^k T^*\mathrm{M}) \times L^2(\Lambda^k T^*\mathrm{M})$ for every $k \in \mathbb{N}$.

As in the smooth setting we have

$$\omega \in \mathrm{Harm}_k(\mathrm{M}) \quad \Longleftrightarrow \quad \omega \in H^{1,2}_H(\Lambda^k T^*\mathrm{M}) \text{ with } \mathrm{d}\omega = 0 \text{ and } \delta\omega = 0. \tag{1.4.7.12}$$

This can be seen noticing that if $\omega \in \mathrm{Harm}_k(\mathrm{M})$, then $\langle \omega, \Delta_H\omega\rangle = 0$ $\mathfrak{m}$-a.e. and so

$$0 = \int \langle \omega, \Delta_H\omega\rangle\,\mathrm{d}\mathfrak{m} \overset{(1.4.7.10)}{=} \int |\mathrm{d}\omega|^2 + |\delta\omega|^2\,\mathrm{d}\mathfrak{m},$$

while the other implication is trivial. Moreover it is useful to know that

$$f \in D(\Delta_{\mathsf{loc}}) \quad \Leftrightarrow \quad \mathrm{d}f \in D(\Delta_{H,\mathsf{loc}}) \qquad \text{and in this case} \qquad \mathrm{d}\Delta f = -\Delta_H \mathrm{d}f \tag{1.4.7.13}$$

where the minus sign is due to the usual sign convention $\Delta f = -\Delta_H f$; this is a direct consequence of the analogous identity valid for objects in $D(\Delta), D(\Delta_H)$ and a cut-off argument.

In the language of vector fields (1.4.7.12) takes the form

$$X^\flat \in D(\Delta_H) \text{ with } \Delta_H X = 0 \quad \Leftrightarrow \quad \begin{cases} X^\flat \in D(\mathrm{d}),\ X \in D(\mathrm{div}) \\ \text{with } \mathrm{d}X^\flat = 0,\ \mathrm{div}X = 0 \end{cases} \tag{1.4.7.14}$$

Finally we point out that the closure of the Hodge Laplacian ensures that $\mathrm{Harm}_k(\mathrm{M})$ is a closed subspace of $L^2(\Lambda^k T^*\mathrm{M})$ and thus, in particular, an Hilbert space itself when endowed with the $L^2(\Lambda^k T^*\mathrm{M})$-norm. Therefore we have the following result:

Theorem 1.4.35 (Hodge theorem on RCD spaces)**.** *For any $k \in \mathbb{N}$ the map*

$$Harm_k(\mathrm{M}) \ni \omega \quad \mapsto \quad [\omega] \in \mathcal{H}^k_{dR}(\mathrm{M})$$

is an isomorphism of Hilbert spaces.

proof We start recalling that if V is a subspace of an Hilbert space H and $V^\perp$ is its orthogonal subspace in H, then the map

$$V^\perp \ni w \quad \mapsto \quad w + \overline{V} \in H/\overline{V}$$

is an isomorphism of Hilbert spaces.

Therefore if we apply this statement to the Hilbert space $C_k(\mathrm{M})$ endowed with the $L^2(\Lambda^k T^*\mathrm{M})$-norm. As for the subspace V we take $\overline{E}_k(\mathrm{M})$ and we notice that the definition of codifferential grants that $\omega \in \mathrm{D}(\delta_k)$ with $\delta\omega = 0$ if and only if ω is orthogonal to $\mathrm{E}_k(\mathrm{M})$. Hence we can conclude by (1.4.7.12) just recalling that $\mathrm{d}\omega = 0$ for every $\omega \in C_k(\mathrm{M})$. □

It is worth to underline that the definition of domain of the Hodge Laplacian in terms of forms in $H^{1,2}_H(\Lambda^k T^*\mathrm{M})$, which in particular implies that also the set of harmonic forms is contained in $H^{1,2}_H(\Lambda^k T^*\mathrm{M})$, has been tailored in order to obtain this version of Hodge theorem. Indeed in general there might be forms in $W^{1,2}_H(\Lambda^k T^*\mathrm{M}) \setminus H^{1,2}_H(\Lambda^k T^*\mathrm{M})$ with zero differential and codifferential.

1.4.8 Ricci curvature

In this section we are going to reformulate the crucial Lemma 1.4.12 in terms of the differential calculus developed so far. This in particular allows to define the notion of Ricci curvature tensor on $\mathsf{RCD}(K,\infty)$ spaces.

Lemma 1.4.36 (Reformulation of key Lemma 1.4.12)**.** *Let* $X \in \mathrm{Test}\,V(\mathrm{M})$. *Then* $X^\flat \in D(\Delta_{H,1})$, $|X|^2 \in \mathrm{D}(\boldsymbol{\Delta})$ *and the following inequality holds:*

$$\boldsymbol{\Delta}\frac{|X|^2}{2} \le \Big(|\nabla X|^2_{HS} - \langle X, (\Delta_H X^\flat)^\sharp\rangle + K|X|^2\Big)\,\mathfrak{m}. \tag{1.4.8.1}$$

Remark 1.4.37. For any $X \in \mathrm{Test}\,V(\mathrm{M})$ we have

$$\boldsymbol{\Delta}\frac{|X|^2}{2}(\mathrm{M}) = 0. \tag{1.4.8.2}$$

Indeed if $(\bar{\varphi}_n)$ is a sequence of uniformly Lipschitz functions, which are also uniformly bounded with bounded support and everywhere converging to the function identically equal to 1, then $\boldsymbol{\Delta}\frac{|X|^2}{2}(\mathrm{M})$ is the limit of $\int \bar{\varphi}_n\,\mathrm{d}\boldsymbol{\Delta}\frac{|X|^2}{2}(\mathrm{M}) = -\int \nabla X : (\nabla\varphi_n \otimes X)\,\mathrm{d}\mathfrak{m}$ and $|\nabla X : (\nabla\varphi_n \otimes X)| \le |\nabla X|_{\mathrm{HS}}|X||\nabla\varphi_n|$. Hence, since $|\nabla X|_{\mathrm{HS}}|X| \in L^1(\mathfrak{m})$, we can conclude by the dominated convergence theorem.

For the later discussion it is convenient to read the space $H^{1,2}_H(T^*\mathrm{M})$ in terms of vector fields rather than covector ones. Thus we introduce the definition:

Definition 1.4.38 (The space $H^{1,2}_H(T\mathrm{M})$)**.** *The space* $H^{1,2}_H(T\mathrm{M}) \subset L^2(T\mathrm{M})$ *is the space of vector fields such that* $X^\flat \in H^{1,2}_H(T^*\mathrm{M})$ *equipped with the norm* $\|X\|_{H^{1,2}_H(T\mathrm{M})} := \|X^\flat\|_{H^{1,2}_H(T^*\mathrm{M})}$.

Lemma 1.4.36 gives then the following result:

Corollary 1.4.39. *We have* $H^{1,2}_H(T\mathrm{M}) \subset H^{1,2}_C(T\mathrm{M})$ *and*

$$\mathsf{E}_C(X) \le \mathsf{E}_H(X^\flat) - \frac{K}{2}\|X\|^2_{L^2(T\mathrm{M})}, \qquad \forall\ X \in H^{1,2}_H(T\mathrm{M}). \tag{1.4.8.3}$$

where E_H *is the Hodge energy defined in* (1.4.7.9)*, while the functional* $\mathsf{E}_C\colon L^2(T\mathrm{M}) \to [0,\infty]$*, called connection energy, is defined as*

$$\mathsf{E}_C(X) := \begin{cases} \dfrac{1}{2}\displaystyle\int |\nabla X|^2_{HS}\,\mathrm{d}\mathfrak{m}, & \text{if } X \in W^{1,2}_C(T\mathrm{M}),\\ +\infty & \text{otherwise.}\end{cases} \tag{1.4.8.4}$$

We remark that for vector fields $X \in \mathrm{D}(\Delta_H)$ the inequality (1.4.8.3) takes the form

$$\int |\nabla X|^2_{\mathrm{HS}}\,\mathrm{d}\mathfrak{m} \le \int \langle X^\flat, \Delta_H X^\flat\rangle - K|X|^2\,\mathrm{d}\mathfrak{m}. \tag{1.4.8.5}$$

An interesting consequence of the inequality (1.4.8.3) is the following bound, which generalizes to this non-smooth setting a classical idea of Bochner, on the dimension of $\mathcal{H}^1_{\mathrm{dR}}(\mathrm{M})$ on spaces with non-negative Ricci curvature:

Proposition 1.4.40 (Bound on $\mathsf{dim}(\mathcal{H}^1_{\mathrm{dR}}(\mathrm{M}))$ on $\mathsf{RCD}(0,\infty)$ spaces)**.** *Let* $(\mathrm{M},\mathsf{d},\mathfrak{m})$ *be an* $\mathsf{RCD}(0,\infty)$ *space and* $(E_i)_{i\in\mathbb{N}\cup\{\infty\}}$ *the dimensional decomposition of* M *associated to the cotangent module* $L^2(T^*\mathrm{M})$ *(see Proposition 1.2.15). We denote by* $\mathsf{dim}_{\min}(\mathrm{M})$ *(resp.* $\mathsf{dim}_{\max}(\mathrm{M})$*) the minimal (resp. supremum) of indexes* $i\in\mathbb{N}\cup\{\infty\}$ *such that* $\mathfrak{m}(E_i)>0$*. Then*

$$\mathsf{dim}(\mathcal{H}^1_{dR}(\mathrm{M}))\leq \mathsf{dim}_{\min}(\mathrm{M}). \tag{1.4.8.6}$$

proof Theorem 1.4.35 ensures that $\mathsf{dim}(\mathcal{H}^1_{\mathrm{dR}}(\mathrm{M}))=\mathsf{dim}(\mathrm{Harm}_1(\mathrm{M}))$. Let us then assume that $\mathsf{dim}(\mathrm{Harm}_1(\mathrm{M}))\geq 1$ (in the case in which $\mathrm{Harm}_1(\mathrm{M})=\{0\}$ there is nothing to prove); directly from (1.4.8.3) we deduce that for every $\omega\in\mathrm{Harm}_1(\mathrm{M})$ it holds

$$\mathsf{E}_C(\omega^\sharp)\leq \mathsf{E}_H(\omega)=\int\langle\omega,\Delta_H\omega\rangle\,\mathrm{d}\mathfrak{m}=0,$$

which in particular means that the covariant derivative of $\omega^\sharp$ is identically 0.

For any two $\omega_1,\omega_2\in\mathrm{Harm}_1(\mathrm{M})$ the function $f:=\langle\omega_1,\omega_2\rangle=\langle\omega_1^\sharp,\omega_2^\sharp\rangle$ is $\mathfrak{m}$-a.e. constant. Indeed, by Proposition 1.4.20, we have that $f\in W^{1,2}(\mathrm{M})$ with $|\mathrm{d}f|=0$ $\mathfrak{m}$-a.e., and we get the conclusion just recalling the Sobolev-to-Lipschitz property (Theorem 1.4.2).

At this point let $\omega_1,\dots,\omega_n\in(\mathrm{Harm}_1(\mathrm{M}),\|\cdot\|_{L^2(T^*\mathrm{M})})$ be an orthonormal set: since $\int\langle\omega_i,\omega_j\rangle\,\mathrm{d}\mathfrak{m}$ is equal to 1 if $i=j$, while it is equal to 0 otherwise, the property above on the $\mathfrak{m}$-a.e. constancy of the function $|\omega_i|^2$ implies that $\mathfrak{m}(X)<\infty$ and

$$|\omega_i|^2=\frac{1}{\mathfrak{m}(X)},\quad \mathfrak{m}\text{-a.e. }\forall\, i\quad\text{and}\quad\langle\omega_i,\omega_j\rangle=0,\quad\mathfrak{m}\text{-a.e. }\forall\, i\neq j.$$

Then the forms $\omega_1,\dots,\omega_n$ are locally independent on M (see Definition 1.2.12) and so they are also on $E_{\mathsf{dim}_{\min}(\mathrm{M})}$. This in particular means that $n\leq\mathsf{dim}_{\min}(\mathrm{M})$. Now we notice that this is true for any choice of n orthonormal elements of $\mathrm{Harm}_1(\mathrm{M})$ which grants that $\mathsf{dim}(\mathrm{Harm}_1(\mathrm{M}))\leq\mathsf{dim}_{\min}(\mathrm{M})$. □

Notice that in the hypothesis of this result we are not assuming the finiteness of the dimension, so we might have $\mathsf{dim}_{\min}(\mathrm{M})=\infty$ and so a priori such result might be empty.

In fact, the natural assumption on the space M is not that it is a $\mathsf{RCD}(0,\infty)$ space, but rather a $\mathsf{RCD}^*(0,N)$ one: given that the number $N\in[1,\infty]$ represents, in some sense, an upper bound for the dimension of the space. We refer to [45] for the proof of the following result:

Theorem 1.4.41. *Let* $(\mathrm{M},\mathsf{d},\mathfrak{m})$ *be a* $\mathsf{RCD}^*(K,N)$ *space. Then* $\mathsf{dim}_{\max}(\mathrm{M})\leq N$.

Therefore coupling Theorems 1.4.40 and 1.4.41 we deduce that:

Proposition 1.4.42. *Let* $(\mathrm{M},\mathsf{d},\mathfrak{m})$ *be a* $\mathsf{RCD}^*(0,N)$ *space. Then* $\dim(\mathcal{H}^1_{dR})(\mathrm{M})\leq N$.

Let us turn to another important consequence of Lemma 1.4.36, that is the definition of Ricci curvature:

Theorem 1.4.43 (Ricci curvature)**.** *There exists a unique continuous map* $\mathbf{Ric}\colon[H^{1,2}_H(T\mathrm{M})]^2\to\mathrm{Meas}(\mathrm{M})$ *such that for every* $X,Y\in\mathrm{Test}\,V(\mathrm{M})$ *it holds:*

$$\mathbf{Ric}(X,Y)=\mathbf{\Delta}\frac{\langle X,Y\rangle}{2}+\left(\frac{1}{2}\langle X,(\Delta_H Y^\flat)^\sharp\rangle+\frac{1}{2}\langle Y,(\Delta_H X^\flat)^\sharp\rangle-\nabla X:\nabla Y\right)\mathfrak{m}. \tag{1.4.8.7}$$

This map, which we call Ricci curvature, is bilinear, symmetric and satisfies:

$$\mathbf{Ric}(X,X)\geq K|X|^2\mathfrak{m}, \tag{1.4.8.8}$$

$$\mathbf{Ric}(X,Y)(\mathrm{M})=\int\langle\mathrm{d}X^\flat,\mathrm{d}Y^\flat\rangle+\delta X^\flat\;\delta Y^\flat-\nabla X:\nabla Y\,\mathrm{d}\mathfrak{m}, \tag{1.4.8.9}$$

$$\|\mathbf{Ric}(X,Y)\|_{TV}\leq 2\sqrt{\mathsf{E}_H(X^\flat)-K^-\|X\|^2_{L^2(T\mathrm{M})}}\;\sqrt{\mathsf{E}_H(Y^\flat)-K^-\|Y\|^2_{L^2(T\mathrm{M})}} \tag{1.4.8.10}$$

for every $X, Y \in H^{1,2}_H(T\mathrm{M})$*, where* $K^- := \max\{0, -K\}$.

We notice that in the case in which $X = Y \in \mathrm{Test}\,V(\mathrm{M})$ (1.4.8.7) gives the usual Bochner identity

$$\mathbf{\Delta}\frac{|X|^2}{2} = \Big(|\nabla X|^2_{\mathrm{HS}} - \langle X, (\Delta_H X^\flat)^\sharp\rangle\Big)\,\mathfrak{m} + \mathbf{Ric}(X, X)$$

and in particular if $X = \nabla f$ for some $f \in \mathrm{Test}F(\mathrm{M})$ we have

$$\mathbf{\Delta}\frac{|\nabla f|^2}{2} = \Big(|\mathrm{Hess}(f)|^2_{\mathrm{HS}} - \langle \nabla f, \nabla \Delta f\rangle\Big)\,\mathfrak{m} + \mathbf{Ric}(\nabla f, \nabla f)$$

Moreover some algebraic computations show that for every $X, Y \in H^{1,2}_H(T\mathrm{M})$ and every $f \in \mathrm{Test}F(\mathrm{M})$ it holds

$$\mathbf{Ric}(fX, Y) = f\mathbf{Ric}(X, Y),$$

which is a tensor-like property for the Ricci curvature map.

A very important consequence of Definition 1.4.43 is the fact that to have at disposal a bound on the Ricci curvature tensor allows to generalize the Bakry-Émery contraction estimate for 1-forms.

For that purpose we first introduce the heat flow $(\mathsf{h}_{H,t})$ on 1-forms as the gradient flow of the augmented Hodge energy functional $\tilde{\mathsf{E}}_H\colon L^2(T^*\mathrm{M}) \to [0, \infty]$ defined in (1.4.7.11): for every $\omega \in L^2(T^*\mathrm{M})$ the curve $t \mapsto \mathsf{h}_{H,t}(\omega) \in L^2(T^*\mathrm{M})$ is the unique continuous curve on $[0, \infty)$ which is locally absolutely continuous on $(0, \infty)$ and satisfies $\mathsf{h}_{H,t}(\omega) \in \mathrm{D}(\Delta_{H,1})$ with

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{h}_{H,t}(\omega) = -\Delta_H \mathsf{h}_{H,t}(\omega), \qquad \forall\, t > 0. \tag{1.4.8.11}$$

In particular for $f \in W^{1,2}(\mathrm{M})$ using the uniqueness of heat flow we see that

$$\mathsf{h}_{H,t}(\mathrm{d}f) = \mathrm{d}\mathsf{h}_t(f), \qquad \forall\, t \geq 0. \tag{1.4.8.12}$$

Therefore we have the following estimate which generalizes the Bakry-Émery contraction estimate for 1-forms:

Proposition 1.4.44. *For every* $\omega \in L^2(T^*\mathrm{M})$ *we have*

$$|\mathsf{h}_{H,t}(\omega)|^2 \leq e^{-2Kt}\mathsf{h}_t(|\omega|^2), \quad \mathfrak{m}\text{-}a.e., \quad \forall\, t \geq 0. \tag{1.4.8.13}$$

Finally we observe that the definition of Ricci curvature tensor on $\mathsf{RCD}(K, \infty)$ is given in such a way that an assumption on the weak curvature is sufficient to find also a bound from below on the Ricci curvature. A natural question is then whether the viceversa holds, namely if a lower bound on the Ricci curvature tensor gives any information in terms of synthetic approach of lower Ricci curvature bounds. A first result in this direction, which can be proved directly from the characterization of $\mathsf{RCD}(K, \infty)$ spaces through the Bochner inequality (we refer to [10] for a proof of it), is the following:

$$\left.\begin{array}{c}\text{let } K' > K \text{ and } (\mathrm{M}, \mathsf{d}, \mathfrak{m}) \text{ be a } \mathsf{RCD}(K, \infty) \text{ space}\\ \text{with } \mathbf{Ric}(X, X) \geq K'|X|^2\mathfrak{m}\ \forall X \in H^{1,2}_H(T\mathrm{M})\end{array}\right\} \implies (\mathrm{M}, \mathsf{d}, \mathfrak{m}) \text{ is a } \mathsf{RCD}(K', \infty) \text{ space.}$$

CHAPTER 2

Regular Lagrangian Flows

2.1 Overview of the chapter

In this chapter we see how the notion of 'Regular Lagrangian flow' developed by Ambrosio in [2] in connection with the Di Perna-Lions theory [30] can be adapted to RCD spaces. For this purpose we follow the works [15] and [16] by Ambrosio and Trevisan, tailoring this approach to the language and the constructions proposed by Gigli in [36]: indeed the theory explained in [15] is set in the more abstract framework of topological spaces, which is the typical one of Γ-calculus and of the theory of Dirichlet forms. In this context the distance is absent, the space M is equipped just with a topology τ and a Borel, non negative and σ-finite reference measure $\mathfrak{m}$; on $L^2(\mathfrak{m})$ it is given a symmetric, densely defined and strongly local Dirichlet form, whose semigroup is assumed to be Markovian, and which originates in its domain a Carré du Champ. Moreover the 'vector fields' are defined as derivations acting on the set of Sobolev functions; in particular the Carré du Champ provides, by duality, a natural pointwise norm on derivations and such duality can be used to define, via integration by parts, a notion of divergence for a derivation.

What we are going to present is a specialization of this general theory to the setting of $\mathsf{RCD}(K,\infty)$ spaces. In particular the theory explained in Chapter 1 provides all the necessary vocabulary to speak about Sobolev functions, Sobolev vector fields and their divergence, as well as all the tools of the theory of heat flow.

Before explaining the structure of this chapter, let us shortly remind the basic facts on Cauchy-Lipschitz theory. We know that the method of characteristics ensures that the flow map Fl gives also the unique solutions to the continuity equation and the transport equation. We recall that the continuity equations takes the conservative form

$$\frac{\mathrm{d}}{\mathrm{d}t}u_t + \mathrm{div}(X_t u_t) = 0 \tag{2.1.0.1}$$

and that the solution, in measure-theoretic therms, is provided by

$$u_t\, \mathscr{L}^n = \mathrm{Fl}(t,\cdot)_\sharp(\bar{u}\mathscr{L}^n). \tag{2.1.0.2}$$

Instead the transport equation

$$\frac{\mathrm{d}}{\mathrm{d}t}w_t + X_t \cdot \nabla w_t = c_t w_t \tag{2.1.0.3}$$

can be solved by integrating the linear ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}w_t(\mathrm{Fl}(t,x)) = c_t(\mathrm{Fl}(t,x))w_t(\mathrm{Fl}(t,x)).$$

In the abstract setup of metric measure spaces the continuity equation is more useful than the transport equation, that is why the work [15] is focussed mostly on it, even though the techniques developed here can be used to obtain well-posedness results also for general transport equations. Indeed in [40] it has been proved that it makes sense to write the continuity equation on a metric measure space $(\mathrm{M}, \mathsf{d}, \mathfrak{m})$ and that absolutely continuous curves (μ_t) with respect to the distance W_2 can be completely characterized as solutions of the continuity equation itself, once the condition $\mu_t \leq C\mathfrak{m}$ has been imposed for every $t \geq 0$ and some $C > 0$. Moreover we notice that, in the Euclidean setting, the continuity equation thanks to its conservative form has the following weak formulation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} f u_t \,\mathrm{d}x = \int_{\mathbb{R}^n} X_t \cdot \nabla f u_t \,\mathrm{d}x, \qquad \text{in } \mathcal{D}'(0,T)$$

for all $f \in C^1_c(\mathbb{R}^n)$. Instead, to give a weak formulation of (2.1.0.3) we need to know that $\mathrm{div}X_t$ is a function and then write it in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}w_t + \mathrm{div}(X_t w_t) = (c_t + \mathrm{div}X_t)w_t,$$

which, in the weak formulation, becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} f w_t \,\mathrm{d}x = \int_{\mathbb{R}^n} (X_t \cdot \nabla f w_t + (c_t + \mathrm{div}X_t)w_t f) \,\mathrm{d}x, \qquad \text{in } \mathcal{D}'(0,T)$$

for all $f \in C^1_c(\mathbb{R}^n)$. In particular we point out the equivalence of the two equations when $c_t = -\mathrm{div}X_t$.

Then, in the first section of this chapter we prove existence of solutions to the weak formulation of the continuity equation (2.1.0.1) defined in a $\mathsf{RCD}(K,\infty)$ space $(\mathrm{M}, \mathsf{d}, \mathfrak{m})$ and induced by a family of vector fields (X_t), namely

$$\frac{\mathrm{d}}{\mathrm{d}t}\int f u_t \,\mathrm{d}\mathfrak{m} = \int \mathrm{d}f(X_t) u_t \,\mathrm{d}\mathfrak{m}, \qquad \forall\, f \in W^{1,2}(\mathrm{M}).$$

In order to do it no regularity on the vector fields X_t is required, but some growth bounds on X_t and on its divergence are needed, in particular a L^∞ lower bounds on $\mathrm{div}X_t$ plays a crucial role for the validity of the argument, as noticed in [2]. The proof of this result closely follows the one in the Euclidean setting: first of all we add a viscosity term and we obtain a solution in $L^2((0,T), W^{1,2}(\mathrm{M}))$ by Hilbert space techniques. Hence the fact that these solutions are uniformly bounded in $L^\infty(L^1)$ in the viscosity parameter allows to take a vanishing viscosity limit. We underline that with this argument we recover not only the existence of the solution to the continuity equation, but also higher integrability estimates on it, depending on the initial condition. Moreover in Proposition 2.2.1 we prove that the L^1-norm of the solution is independent on time.

In Section 2.2.2 we prove the uniqueness for the solution of the continuity equation (that is strictly linked to the unicity of the Regular Lagrangian flow, as we are going to prove). We start noticing that for this aim the Sobolev regularity of the vector fields (X_t) plays a fundamental role: without this assumption, uniqueness may fail even for divergence-free vector fields (see for instance [29]). It turns out that, with this hypothesis on the regularity of (X_t), the weak solutions to the continuity equation satisfy a crucial regularity property:

Definition 2.1.1 (Renormalized solution)**.** *We say that a weak solution $u \in L^\infty(L^1 \cap L^\infty)$ to the continuity equation is renormalized if*

$$\frac{\mathrm{d}}{\mathrm{d}t}\beta(u_t) + X_t \cdot \nabla\beta(u_t) = -u_t\beta'(u_t)\mathrm{div}X_t \tag{2.1.0.4}$$

holds still in the weak sense, for all $\beta \in C^1(\mathbb{R})$.

Indeed, in [30] Di Perna and Lions proved that once we know that a solution to (2.1.0.1) is renormalized, then the continuity equation with initial datum $\bar{u} \in L^2(\mathfrak{m})$ is well-posed in $L^\infty(L^1 \cap L^\infty)$. Also in this setting we argue in this way: we prove that the solution obtained in Section 2.2.1 are renormalized solution. We do it by regularization, namely writing $u_t^\alpha = \mathsf{h}_\alpha u_t$ and looking at the PDE satisfied by $t \mapsto u_t^\alpha$, that is

$$\frac{\mathrm{d}}{\mathrm{d}t} u_t^\alpha + X_t \cdot \nabla u_t^\alpha = -u_t^\alpha \mathrm{div} X_t + \mathscr{C}^\alpha(X_t, u_t), \tag{2.1.0.5}$$

where $\mathscr{C}^\alpha(Y, v)$ is defined by

$$\mathscr{C}^\alpha(Y, v) := \mathrm{div}((\mathsf{h}_\alpha v)Y) - \mathsf{h}_\alpha^*(\mathrm{div}(vY)).$$

We have used the notation h_α^* because, strictly speaking, $\mathrm{div}(vY)$ is not a function, so the action of the heat flow should be understood in the dual sense.

Therefore Section 2.2.3 is devoted to the proof of the fact that

$$\lim_{\alpha \to 0} \int_0^T \|\mathscr{C}^\alpha(X_t, u_t)\|_1 \, \mathrm{d}t = 0,$$

provided we have some bound on the symmetric part of the derivative (Definition 2.2.3.7), necessary to give an estimate of the commutator. In particular in our setting such a bound follows by giving a bound on the norm of the covariant derivative of the vector fields (X_t), as we see in Remark 2.2.9.

In Section 2.3 we pass to the Lagrangian side of the theory: we see how the definition of ODE and Regular Lagrangian flow can be adapted to the setting of metric measure spaces. In particular we show how solutions u to the continuity equation $\partial_t u_t + \mathrm{div}(u_t X_t) = 0$ can be lifted to measures $\boldsymbol{\eta}$ in $C([0,T]; \mathrm{M})$: we will see that $(e_t)_\sharp \boldsymbol{\eta} = u_t \mathfrak{m}$ for all $t \in (0,T)$ and that $\boldsymbol{\eta}$ is concentrated on solutions η to the ODE $\dot{\eta} = X_t(\eta)$. The first tool that allows to pass from the Lagrangian to the Eulerian point of view is given by Lemma 2.3.4, which shows that time marginals of measures $\boldsymbol{\eta}$ concentrated on solutions to the ODE $\dot{\eta} = X_t(\eta)$ provide weakly continuous solutions to the continuity equation.

The converse mechanism (i.e., the tool that allows to pass from the Eulerian perspective to the Lagrangian one) is given by the Superposition Principle, presented in Section 2.4. Thanks to this result, starting from the continuity equation, we can find a plan $\boldsymbol{\eta} \in \mathscr{P}(C([0,T]; \mathrm{M}))$ that is concentrated on curves η satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}(f \circ \eta) = \mathrm{d}f(X_t) \circ \eta$$

in the sense of distributions in $(0,T)$, for all $f \in W^{1,2}(\mathrm{M})$, which is the natural generalization of solution of the ODE $\dot{\eta} = X_t(\eta)$ to our setting of RCD spaces. Moreover we show that this property implies absolutely continuity of $\boldsymbol{\eta}$-almost every curve η, with the metric derivative $|\dot{\eta}|$ estimated from above by $|X_t| \circ \eta$.

Finally in Section 2.5 we are in the position to prove Theorem 2.5.2, that links the well-posedness of the continuity equation in the class of nonnegative functions $L_t^1(L_x^1 \cap L_x^\infty)$ with initial data $\bar{u} \in L^1 \cap L^\infty(\mathfrak{m})$ to the existence and uniqueness of the flow Fl, according to Definition 2.3.3. The proof of this result is based on two facts: first, the possibility to lift the solution u to a plan $\boldsymbol{\eta}$, given by the Superposition Principle; then, the fact that the restriction of $\boldsymbol{\eta}$ to any Borel set still induces a solution to the continuity equation with the same velocity field. Hence in particular we can localize $\boldsymbol{\eta}$ to show that there is non-uniqueness at the level of continuity equation whenever some branching of trajectories occurs (Theorem 2.5.4).

We conclude this chapter giving a characterization of Regular Lagrangian flows in the case in which the family of vector fields (X_t) satisfies $|X_t| \in L^\infty([0,T], L^\infty(\mathrm{M}))$ (see Proposition 2.5.7).

Notation: We define the set of functions

$$\mathcal{L} := \{f\colon \mathrm{M} \to \mathbb{R} : f \text{ is a Lipschitz function with bounded support}\}$$

and for $p \in [1,\infty)$ the spaces

$$\mathbb{V}_p := \left\{ f \in W^{1,2}(\mathrm{M}) \cap L^p(\mathfrak{m}) : \int |\mathrm{D}f|^p \, \mathrm{d}\mathfrak{m} < \infty \right\},$$

with the obvious extension to $p = \infty$. Moreover we endow each $\mathbb{V}_p$ with the norm

$$\|f\|_{\mathbb{V}_p} = \|f\|_{W^{1,2}(\mathrm{M})} + \|f\|_p + \||\mathrm{D}f|\|_p, \tag{2.1.0.6}$$

thus obtaining a Banach space. We notice that $\mathbb{V}_2 = W^{1,2}(\mathrm{M})$, with an equivalent norm.

Finally we point out that $\mathcal{L}$ is a dense in $W^{1,2}(\mathrm{M})$ and contained in $L^p(\mathfrak{m})$ for any $p \in [1,\infty]$.

2.2 Eulerian point of view

2.2.1 Existence of solutions to the continuity equation

In this section we show the existence of weak solutions to the continuity equation

$$\partial_t u_t + \mathrm{div}(X_t u_t) = 0, \quad \text{in } [0,T] \times \mathrm{M} \tag{2.2.1.1}$$

for a family of vector fields $(X_t) \in L^1([0,T], L^2_{\mathsf{loc}}(T\mathrm{M}))$, with the initial datum $u_0 = \bar{u} \in L^\infty_t(L^2_x)$. Here by weak solution we refer to the solution of the weak formulation of equation (2.2.1.1), namely we are requiring that for any $f \in W^{1,2}(\mathrm{M})$ the function $t \mapsto \int f u_t \, \mathrm{d}\mathfrak{m}$ is absolutely continuous in $[0,T]$ and that its a.e. derivative in $[0,T]$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} \int u_t f \, \mathrm{d}\mathfrak{m} = \int \mathrm{d}f(X_t) u_t \, \mathrm{d}\mathfrak{m} \quad \text{in } \mathcal{D}'(0,T),$$

while the Cauchy initial condition is expressed by requiring that $\int u_t f \, \mathrm{d}\mathfrak{m} \to \int \bar{u} f \, \mathrm{d}\mathfrak{m}$ as $t \downarrow 0$ for all $f \in W^{1,2}(\mathrm{M})$.

We point out that in this weak formulation u_t is determined up to a $\mathscr{L}^1$-negligible set of times. However, using the fact that $t \mapsto \int u_t f \, \mathrm{d}\mathfrak{m}$ has a continuous representative for all $f \in W^{1,2}(\mathrm{M})$, it follows that there exists a continuous representative $t \mapsto u_t$ which is continuous in duality with elements in $W^{1,2}(\mathrm{M})$. In the sequel we will refer to this continuous representative.

Proposition 2.2.1. *Let $\bar{u} \in L^2(\mathfrak{m})$ and assume that $(X_t) \in L^2([0,T], L^2(T\mathrm{M}))$. Then any weak solution $(u_t) \in L^2_t(L^2_x)$ satisfies the mass conservation property*

$$\int u_t \, \mathrm{d}\mathfrak{m} = \int \bar{u} \, \mathrm{d}\mathfrak{m}, \qquad \forall t \in [0,T]. \tag{2.2.1.2}$$

proof In the case in which the measure of the space is finite, (2.2.1.2) follows just observing that $1 \in W^{1,2}(\mathrm{M})$. Otherwise, if $\mathfrak{m}(\mathrm{M}) = \infty$, we have to use an approximation argument. A standard cut-off procedure ensures that there exists a sequence $(f_n) \subset \mathcal{L}$ of Lipschitz functions with bounded support such that $0 \le f_n \le 1$, $f_n \uparrow 1$ $\mathfrak{m}$ in M and with $|\mathrm{D}f_n| \rightharpoonup 0$ weakly-$*$ in $L^\infty(\mathfrak{m})$.

We observe that (2.2.1.2) is equivalent to prove that the two measures $\mu_0 = u_0\mathfrak{m} = \bar{u}\mathfrak{m}$ and $\mu_T = u_T\mathfrak{m}$ have the same mass. In order to do it we start observing that

$$\mu_T(\mathrm{M}) - \mu_0(\mathrm{M}) = \lim_{n\to\infty} \int f_n \,\mathrm{d}\mu_T - \int f_n \,\mathrm{d}\mu_0.$$

Hence, since $f_n \in W^{1,2}(\mathrm{M})$, for any fixed $n \in \mathbb{N}$ the continuity equation ensures that

$$\int f_n \,\mathrm{d}\mu_T - \int f_n \,\mathrm{d}\mu_0 = \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} \int f_n u_t \,\mathrm{d}\mathfrak{m}\,\mathrm{d}t = \int_0^T \int \mathrm{d}f_n(X_t) u_t \,\mathrm{d}\mathfrak{m}.$$

On the other hand, directly from the properties of the sequence (f_n) it follows that $\int |\mathrm{d}f_n(X_t)| u_t \,\mathrm{d}\mathfrak{m}$ goes to 0 as $n \to \infty$ for any fixed $t \in [0,T]$. Now we remark that, since the sequence of differentials $\{|\mathrm{d}f_n|\}_{n\in\mathbb{N}}$ is uniformly bounded and $|X_t|u_t \in L^1_t(L^1_x)$, the dominated convergence theorem ensures that

$$\lim_{n\to\infty} \int_0^T \int |\mathrm{d}f_n(X_t)| u_t \,\mathrm{d}\mathfrak{m}\,\mathrm{d}t = 0$$

which in turn implies that $\int u_t \,\mathrm{d}\mathfrak{m} = \int \bar{u} \,\mathrm{d}\mathfrak{m}$ for all $t \in [0,T]$. □

Theorem 2.2.2. *Let $\bar{u} \in L^2(\mathfrak{m})$. Assume that $(X_t) \in L^1([0,T], L^2(T\mathrm{M}))$ with $\mathrm{div}(X_t) \in L^1([0,T], L^2(\mathfrak{m}))$ and $(\mathrm{div}(X_t))^- \in L^1([0,T], L^\infty(\mathfrak{m}))$. Then there exists $u \in L^\infty([0,T], L^2(\mathfrak{m}))$ solution to* (2.2.1.1)*.*

For the proof of this theorem we use the classical method of vanishing viscosity solutions: first of all we prove the existence of a solution to a regularized equation, then we prove a priori estimates in $L^2(\mathfrak{m})$ and finally we take the limit as the viscosity parameter goes to 0.

Step 1: Existence of the solution for the regularized equation. For $\sigma \in (0,1)$ we want to prove the existence of a solution u^σ for the equation

$$\partial_t u_t + \mathrm{div}(X_t u_t) = \sigma\Delta u_t, \qquad u_0 = \bar{u}. \tag{2.2.1.3}$$

In order to do it a key result we are going to use is Lions' lemma, which is a generalization of Lax-Milgram theorem:

Lemma 2.2.3 (Lions)**.** *Let H be a Hilbert space and V a normed space continuously embedded in H, with $\|v\|_H \le \|v\|_V$ for all $v \in V$. Suppose that the bilinear map $B\colon V \times H \to \mathbb{R}$ is coercive, i.e., there exists $c > 0$ such that $B(v,v) \ge c\|v\|_V^2$ for all $v \in V$, and such that $B(v,\cdot)$ is continuous for any $v \in V$. Then for all $\ell \in V'$ there exists $h \in H$ such that $B(\cdot,h) = \ell$ and*

$$\|h\|_H \le \frac{\|\ell\|_{V'}}{c}.$$

We apply this lemma in the case in which $H = L^2((0,T), W^{1,2}(\mathrm{M}))$ and $V \subset H$ is the vector space generated by the functions ψf, where $\psi \in C^1_c([0,T))$ and f is a Lipschitz function on M with bounded support; we endow V with the norm

$$\|\varphi\|_V^2 = \|\varphi\|_H^2 + \|\varphi_0\|_2^2.$$

The bilinear form we consider is given by

$$B(\varphi, h) := \int_0^1 \int [-\partial_t \varphi + \lambda\varphi - \mathrm{d}\varphi(X_t)]h + \sigma \mathrm{d}\varphi(\nabla h) \,\mathrm{d}\mathfrak{m}\,\mathrm{d}t,$$

while $\ell(\varphi) := \int \varphi_0 \bar{u} \,\mathrm{d}\mathfrak{m}$. Directly from the definition of norm in V we see that $\ell \in V'$ and that $\|\ell\|_{V'} \leq \|\bar{u}\|_2$, while the fact that $(X_t) \in L^\infty_t(L^2(\mathfrak{m}))$ ensures that $B(\varphi, \cdot)$ is continuous in H for any $\varphi \in V$. The only thing left to prove is the coercivity of B for a suitable choice of λ: for any $\varphi \in V$ we have

$$\int_0^1 \int [-\partial_t \varphi + \lambda\varphi \mathrm{d}\varphi(X_t)]\varphi \,\mathrm{d}\mathfrak{m}\,\mathrm{d}t = \int_0^1 \int -\frac{1}{2}\partial_t \varphi_t^2 - \frac{1}{2}\mathrm{d}\varphi^2(X_t)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}t + \lambda \|\varphi\|^2_{L^1_t(L^2_x)}.$$

Then Fubini theorem and the fact that $\varphi \in V$ ensure that $\int_0^1 \int \partial_t \varphi_t \,\mathrm{d}\mathfrak{m}\,\mathrm{d}t = \int \varphi_0^2 \,\mathrm{d}\mathfrak{m}$ and that $-\int_0^1 \int \mathrm{d}\varphi^2(X_t)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}t \geq -\int_0^1 \int \varphi^2 \mathrm{div}(X_t)^- \,\mathrm{d}\mathfrak{m}\,\mathrm{d}t$; so we obtain

$$\begin{aligned}\int_0^1 \int [-\partial_t \varphi + \lambda\varphi \mathrm{d}\varphi(X_t)]\varphi \,\mathrm{d}\mathfrak{m}\,\mathrm{d}t &\geq \|\varphi_0\|_2^2 + \left(\lambda - \frac{1}{2}\left\|\mathrm{div}(X_t)^-\right\|_\infty\right)\|\varphi\|^2_{L^2_t(L^2_x)} \\ &\geq \sigma\left(\|\varphi_0\|_2^2 + \|\varphi\|^2_{L^2_t(L^2_x)}\right)\end{aligned}$$

provided that $\lambda \geq \lambda_\sigma = \frac{1}{2}\left\|\mathrm{div}(X_t)^-\right\|_\infty + \sigma$. At this point, taking account of the presence of the term $\sigma \int_0^1 \int |Df|^2 \,\mathrm{d}\mathfrak{m}\,\mathrm{d}t$ in the definition of B, we have

$$B(\varphi, \varphi) \geq \sigma\left(\|\varphi_0\|_2^2 + \|\varphi\|^2_{L^2_t(L^2_x)} + \|D\varphi\|^2_{L^2_t(L^2_x)}\right) = \sigma \|\varphi\|^2_V.$$

Hence we are in the hypothesis needed to apply Lemma 2.2.3 with $\lambda = \lambda_\sigma$ and we obtain a weak solution h to

$$\frac{\mathrm{d}}{\mathrm{d}t}h_t + \mathrm{div}(X_t h_t) + \lambda_\sigma h_t = \sigma h_t, \qquad h_0 = \bar{u}.$$

as well as a bound on the norm given by $\|h\|_H \leq \|\bar{u}\|_2/\sigma$. Then, setting $u^\sigma := e^{\lambda_\sigma t}h$, we find a solution to (2.2.1.3) that satisfies the bound

$$\left\|e^{-\lambda_\sigma t}u^\sigma\right\|_H \leq \frac{\|\bar{u}\|_2}{\sigma}. \tag{2.2.1.4}$$

Step 2: A priori estimates. We want to prove that the approximate solutions u^σ found in Step 1 satisfy the estimate

$$\sup_{t\in(0,T)} \left\|(u^\sigma_t)^\pm\right\|_2 \leq \left\|\bar{u}^\pm\right\|_2 \exp\left(\frac{1}{2}\left\|(\mathrm{div}X_t)^-\right\|_{L^1(L^\infty)}\right). \tag{2.2.1.5}$$

Remark 2.2.4. The proof of this fact strongly relies on the formal identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \beta(u_t)\,\mathrm{d}\mathfrak{m} = -\int [\beta'(u_t)u_t - \beta(u_t)]\,\mathrm{div}X_t \,\mathrm{d}\mathfrak{m}, \tag{2.2.1.6}$$

which comes from the chain rule and the identity $\int \mathrm{div}(\beta(u_t)X_t)\,\mathrm{d}\mathfrak{m} = 0$ (which is again formal).

Let us briefly see a sketch of the proof of the bound in (2.2.1.5). First of all we want to give a meaning to the equality in 2.2.1.6 in the case in which $\beta \colon \mathbb{R} \to [0, \infty)$ is any convex function such that $\beta(0) = 0$ and $\beta'(z)/z$ is bounded on $\mathbb{R}$ (observe that in general β is not C^1); indeed a typical choice for β is given by $z \mapsto z^+$. For that purpose we define the function

$$\mathcal{L}_\beta(z) := \begin{cases} z\beta'_+(z) - \beta(z) & \text{if } z \geq 0, \\ z\beta'_-(z) - \beta(z) & \text{if } z \leq 0, \end{cases} \tag{2.2.1.7}$$

where $\beta'_\pm(z) := \lim_{y\to z^\pm} \beta'(y)$. The assumption on the behaviour of β near the origin guarantees that both $\beta(u)$ and $\mathcal{L}_\beta(u)$ are in $L^2_t(L^1_x)$ for any $u \in L^2_t(L^2_x)$.

It is useful also to approximate β in two different ways: firstly with functions with linear growth in $\mathbb{R}$, and to this aim we consider the sequence given by

$$\beta_n(z) := \begin{cases} \beta(-n) + \beta'_-(-n)(z+n), & \text{if } z < -n; \\ \beta(z) & \text{if } -n \le z \le n; \\ \beta(n) + \beta'_+(n)(z-n), & \text{if } z > n. \end{cases} \tag{2.2.1.8}$$

In particular we observe that $\mathcal{L}_{\beta_n} = \mathcal{L}_\beta(-n \vee z \wedge n)$ and $\mathcal{L}_{\beta_n} \uparrow \mathcal{L}_\beta$ as $n \to \infty$.
Then we want to pass from smooth to non smooth β's and in order to do it we remark that if $\{\beta_i\}_i$ is a sequence of convex functions such that $\beta_i \to \beta$ uniformly on compacts sets, then the definition of $\mathcal{L}_{\beta_i}$ ensures that $\limsup_{i\to\infty} \mathcal{L}_{\beta_i} \le \mathcal{L}_\beta$. In particular this means that in the following it is not ristrective to assume that $\beta \in C^1$ with bounded derivative.

The key point in the proof of (2.2.1.5) consists in showing that the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \beta(u_t)\,\mathrm{d}\mathfrak{m} \le \int \mathcal{L}_\beta(u_t)(\mathrm{div} X_t)^-\,\mathrm{d}\mathfrak{m} \tag{2.2.1.9}$$

holds in the sense of distributions in $(0,T)$.

A first technical point in this direction consists in the necessity to have strong differentiability with respect to time: to get it we use the regularization $u^\varepsilon_t = \mathsf{h}_\varepsilon u_t$ and the following remark [65, Prop III.1.1.] for a proof of it.

Remark 2.2.5. Let $f, g \in L^1((0,T); L^1(\mathfrak{m}) + L^2(\mathfrak{m}))$ satisfying $\partial_t f = g$ in the weak sense, meaning that for every $\psi \in C^1_c(0,T)$ and $\phi \in \mathcal{L}$ it holds

$$-\int_0^T \psi'(t)\int \phi(f)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}t = \int_0^T \psi(t)\int \phi(g)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}t,$$

(we recall that $\mathcal{L}$ is dense in $(L^1(\mathfrak{m}) + L^2(\mathfrak{m}))^* = L^2 \cap L^\infty(\mathfrak{m})$). Then f admits a unique absolutely continuous representative from $(0,T)$ to M and this representative is strongly differentiable a.e. in $(0,T)$, with derivative equal to g.

Note that the space $L^1(\mathfrak{m}) + L^2(\mathfrak{m})$ does not have the Radon-Nikodym property and so it might exist an absolutely continuous maps with values in $L^1(\mathfrak{m}) + L^2(\mathfrak{m})$ which is not strongly differentiable a.e. in its domain.

Then a direct computation shows that for every $\varepsilon > 0$, the function $t \mapsto u^\varepsilon_t$ solves

$$\frac{\mathrm{d}}{\mathrm{d}t}u^\varepsilon_t + \mathrm{div}(X_t u^\varepsilon_t) = \sigma\Delta u^\varepsilon_t + \mathscr{C}^\varepsilon_t$$

in the weak sense in duality with elements in $W^{1,2}(\mathrm{M})$. Here $\mathscr{C}^\varepsilon_t = \mathscr{C}^\varepsilon(u_t, X_t)$ is the commutator between the divergence and the action of the semigroup, that is

$$\mathscr{C}^\varepsilon(u_t, X_t) := \mathrm{div}(X_t \mathsf{h}_\varepsilon(u_t)) - \mathsf{h}_\varepsilon(\mathrm{div}(X_t u_t)). \tag{2.2.1.10}$$

In particular, the assumptions on the vector fields X_t ensure that $\mathscr{C}^\varepsilon_t \to 0$ strongly in $L^2_t(L^1_x + L^2_x)$ as $\varepsilon \downarrow 0$.

At this point the inequality in (2.2.1.9) can be achieved by differentiating with respect to t the entropy $\int \beta(\mathsf{h}_\varepsilon u^\sigma_t)\,\mathrm{d}\mathfrak{m}$ and letting $\varepsilon \downarrow 0$, since $\beta'(u^\varepsilon_t)$ are bounded in $L^2_t(L^2_x \cap L^\infty_x)$, uniformly with respect to ε.

Finally, we briefly see how we can derive (2.2.1.5) from (2.2.1.9). Let $\beta(z) = (z^\pm)^2$ and notice that in this case $\mathcal{L}_\beta(z) = \beta(z)$; however we cannot apply directly (2.2.1.9) to such a function β, since $\beta'(z)/z$ is unbounded near 0. An intermediate step consists then in approximating β with the sequence of β_n in (2.2.1.8), so that $\mathcal{L}_{\beta_n} \le \beta_n$ and $\beta_n'(z)/z$ is bounded, and in particular satisfies $\mathcal{L}_{\beta_n}(z) = \mathcal{L}_\beta(z \wedge n)$. At this point it suffices to apply Gronwall's lemma to (2.2.1.9) for $\beta = \beta_n$ and then let $n \to \infty$ to conclude with Fatou's lemma.

Step 3: Limit as $\sigma \downarrow 0$. We want now to pass to the limit as $\sigma \downarrow 0$ in (2.2.1.3), namely, using the weak formulation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int f u_t^\sigma \, \mathrm{d}\mathfrak{m} = \int \mathrm{d}f(X_t) u_t^\sigma \, \mathrm{d}\mathfrak{m} + \sigma \int \mathrm{d}f(\nabla u_t^\sigma), \quad f \in W^{1,2}(\mathrm{M}).$$

The a priori bound in (2.2.1.5) implies that σu^σ is bounded in $L_t^2(W^{1,2}(\mathrm{M}))$; moreover it can be proved that σu^σ strongly converge to 0 in $L_t^2(L_x^2)$. These two information together imply that σu^σ weakly converges to 0 in $L^2(W^{1,2}(\mathrm{M}))$ and

$$\lim_{\sigma \downarrow 0} \sigma \int_0^1 \int \mathrm{d}f(\nabla u_t^\sigma) \, \mathrm{d}\mathfrak{m} \, \mathrm{d}t = 0.$$

In particular this proves that any weak-* limit of u^σ in $L^\infty(L^2)$ as $\sigma \downarrow 0$ is a solution of (2.2.1.1).

2.2.2 Uniqueness of solutions to the continuity equation

We start recalling that on a $\mathsf{RCD}(K,\infty)$ space $\mathcal{L}$ is dense in $\mathbb{V}_4$ and that the inequality in (1.4.2.7) holds for any $p \in [2,\infty]$, and so in particular for $p = 4$.

Theorem 2.2.6. *Let $(X_t) \in L^2([0,T], W^{1,2}_{C,\mathsf{loc}}(T\mathrm{M})) \cap L^\infty([0,T], L^\infty(T\mathrm{M}))$ be such that $X_t \in D(\mathrm{div}_{\mathsf{loc}})$ for a.e. $t \in [0,T]$, with*

$$\int_0^T \||\nabla X_t|\|_{L^2(\mathrm{M})} + \|\mathrm{div}(X_t)\|_{L^2(\mathrm{M})} + \left\|\big(\mathrm{div}(X_t)\big)^-\right\|_{L^\infty(\mathrm{M})} \mathrm{d}t < \infty. \tag{2.2.2.1}$$

Then there exists a unique solution for the continuity equation in (2.2.1.1).

The existence of this solution is guaranteed by Theorem 2.2.2, while in order to prove the uniqueness we need to prove that weak solutions to (2.2.1.1) are renormalized, namely, for every $\beta \in C^1(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$ it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\beta(u_t) + \langle X_t, \nabla\beta(u_t)\rangle = -u_t\beta'(u_t)\mathrm{div}X_t. \tag{2.2.2.2}$$

As before, the first step consists in regularizing u_t, i.e., we consider $u_t^\varepsilon = \mathsf{h}_\varepsilon u_t$ and we look for the PDE satisfied by $t \mapsto u_t^\varepsilon$. A direct computation shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}u_t^\varepsilon + \mathrm{div}(u_t^\varepsilon X_t) = \mathscr{C}^\varepsilon(X_t, u_t), \tag{2.2.2.3}$$

where $\mathscr{C}^\varepsilon(X_t, u_t)$ is the commutator defined in (2.2.1.10). Hence, the only thing left to prove is that

$$\lim_{\varepsilon \downarrow 0} \int_0^1 \|\mathscr{C}^\varepsilon(X_t, u_t)\|_1 \, \mathrm{d}t = 0. \tag{2.2.2.4}$$

Indeed, once we know it we can pass to the limit as $\varepsilon \downarrow 0$ in the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\beta(u_t^\varepsilon) + \langle X_t, \nabla\beta(u_t^\varepsilon)\rangle = -u_t^\varepsilon\beta'(u_t^\varepsilon)\mathrm{div}(X_t) + \beta'(u_t^\varepsilon)\mathscr{C}^\varepsilon(X_t, u_t),$$

which is derived from (2.2.2.3) thanks to the regularity of u_t^ε, in order to obtain (2.2.2.2).

The main result that allows to conclude is the following commutator estimate:

2.2.3 Commutator lemma

First of all we collect some consequences of (1.4.2.1), which provide an approximation of the vector field X via the action of the heat flow h_α.

Proposition 2.2.7. *Let $X \in L^2(TM) \cap L^\infty(\mathfrak{m})$, then*

(i) For every $\alpha \in (0,\infty)$, the map

$$\mathscr{L} \ni f \mapsto \mathrm{d}(\mathsf{h}_\alpha f)(X)$$

extends uniquely to a linear operator $\boldsymbol{B}^\alpha$ from $L^4 \cap L^2(\mathfrak{m})$ to $L^{4/3}(\mathfrak{m}) + L^2(\mathfrak{m})$ with

$$\|\boldsymbol{B}\| \le C(\alpha \wedge 1)^{-1/2} \|X\|_{L^2+L^\infty} \tag{2.2.3.1}$$

(ii) For all $f \in L^4 \cap L^2(\mathfrak{m})$ the map $\alpha \mapsto \boldsymbol{B}^\alpha(f)$ is continuous from $(0,\infty)$ to $L^{4/3}(\mathfrak{m}) + L^2(\mathfrak{m})$ and, if $\Delta f \in L^4 \cap L^2(\mathfrak{m})$, it is $C^1((0,\infty); L^{4/3}(\mathfrak{m}) + L^2(\mathfrak{m}))$, with

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\boldsymbol{B}^\alpha(f) = \boldsymbol{B}(\Delta f). \tag{2.2.3.2}$$

(iii) In the case in which $u \in L^4 \cap L^2(\mathfrak{m})$ and $X \in \mathrm{D}(\mathrm{div})$ with $\mathrm{div}X \in L^2(\mathfrak{m}) + L^\infty(\mathfrak{m})$. Then

$$\mathrm{div}(\beta(\mathsf{h}_\alpha u)X) = \beta(\mathsf{h}_\alpha u)\mathrm{div}X + \beta'(\mathsf{h}_\alpha u)\boldsymbol{B}^\alpha(u) \in L^{4/3}(\mathfrak{m}) + L^2(\mathfrak{m}) \tag{2.2.3.3}$$

for all $\alpha > 0$ and all $\beta \in C^1(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$ with $\beta(0) = 0$. In particular, for $\beta(z) = z$ (2.2.3.3) gives

$$\mathrm{div}((\mathsf{h}_\alpha u)X) = (\mathsf{h}_\alpha u)\mathrm{div}X + \boldsymbol{B}^\alpha(u) \in L^{4/3}(\mathfrak{m}) + L^2(\mathfrak{m}). \tag{2.2.3.4}$$

(iv) Assume $u \in L^4 \cap L^2(\mathfrak{m})$ and $X \in \mathrm{D}(\mathrm{div})$ with $\mathrm{div}X \in L^2(\mathfrak{m}) + L^\infty(\mathfrak{m})$. Then $\mathscr{C}^\alpha(\mathsf{h}_\delta u, X) \in L^{4/3}(\mathfrak{m}) + L^2(\mathfrak{m})$ for every $\delta > 0$ and

$$\lim_{\alpha\downarrow 0} \|\mathscr{C}^\alpha(\mathsf{h}_\delta u, X)\|_{L^{4/3}+L^2} = 0. \tag{2.2.3.5}$$

proof

(i) First of all we observe that since the vector field X belongs to $L^2(TM) \cap L^\infty(\mathfrak{m})$, we can use the density of $\mathcal{L}$ in $W^{1,2}(\mathrm{M})$ in order to achieve that the map which sends $f \in W^{1,2}(\mathrm{M})$ to $\mathrm{d}f(X) \in L^1(\mathfrak{m}) + L^2(\mathfrak{m})$ is well defined and continuous, and it still satisfies the bound

$$|\mathrm{d}f(X)| \le |X||\mathrm{d}f|, \mathfrak{m}\text{-a.e. in M}.$$

Again , the density of $\mathcal{L}$ in $\mathbb{V}_r$ for any $r \in [2,\infty)$ ensures that also $f \ni \mathbb{V}_r \mapsto \mathrm{d}f(X) \in L^{s'}(\mathfrak{m}) + L^2(\mathfrak{m})$ is a linear continuous map, where $r^{-1} + s^{-1} = 1/2$.

We use this remark with $r = 4$ in order to conclude that $\mathrm{d}(\mathsf{h}_\alpha f)(X)$ is actually well defined.

At this point the validity of (1.4.2.7) for $p = 4$, together with the density of $\mathcal{L}$ in $\mathbb{V}_4$, ensures that for every $f \in \mathcal{L}$ it holds

$$\|\mathrm{d}(\mathsf{h}_t f)(X)\|_{L^4} \le \|X\|_2 \|\mathrm{d}(\mathsf{h}_t f)\|_{L^4} \le c_4(t \wedge 1)^{-1/2}\|X\|_2\|f\|_4.$$

In the same way, if $X \in L^\infty(\mathfrak{m})$ we have

$$\|\mathrm{d}(\mathsf{h}_t f)(X)\|_{L^2} \le \|X\|_\infty \|\mathrm{d}(\mathsf{h}_t f)\|_{L^2} \le c_2(t \wedge 1)^{-1/2}\|X\|_\infty\|f\|_2.$$

Therefore $\|\boldsymbol{B}^\alpha(f)\|_{L^4+L^2} \le \max\{c_4, c_2\}(t \wedge 1)^{-1/2}\|X\|_{L^2+L^\infty}\|f\|_{L^2\cap L^4}$ on $\mathcal{L}$: the density of $\mathcal{L}$ in $L^2 \cap L^4(\mathfrak{m})$ provides the existence of $\boldsymbol{B}^\alpha$ and the estimate on its norm.

(ii) The semigroup law for the heat flow together with the uniqueness of the extension of $\boldsymbol{B}^\alpha$ ensure that

$$\boldsymbol{B}^{\alpha+\sigma}(f) = \boldsymbol{B}^\alpha(\mathsf{h}_\sigma f), \text{ for every } f \in L^2 \cap L^4(\mathfrak{m}),\ \alpha, \sigma \in (0,\infty).$$

Then directly from the bound in (2.2.3.1) and the strong continuity of h_s we obtain

$$\|\boldsymbol{B}^{\alpha+\sigma}(f) - \boldsymbol{B}^\alpha(f)\|_{L^{4/3}(\mathfrak{m})+L^2(\mathfrak{m})} \le \max\{c_4, c_2\}(\alpha \wedge 1)^{-1/2}\|X\|_{L^2+L^\infty}\|\mathsf{h}_\sigma f - f\|_{L^2\cap L^4}$$

and with the same argument we show the differentiability of $\alpha \mapsto \boldsymbol{B}^\alpha(f)$ in the case in which $\Delta f \in L^2 \cap L^4(\mathfrak{m})$.

(iii) We obtain (2.2.3.4) by (1.3.3.6), while (2.2.3.3) follows directly from the chain rule.

(iv) By applying twice (2.2.3.4) we get the identity

$$-\mathscr{C}^\alpha(\mathsf{h}_\delta u, X) = \mathsf{h}_\alpha[(\mathsf{h}_\delta u)\mathrm{divX}] + \mathsf{h}_\alpha(\boldsymbol{B}^\delta(\mathrm{u})) - (\mathsf{h}_{\alpha+\sigma}\mathrm{u})\mathrm{divX} - \boldsymbol{B}^{\alpha+\delta}(\mathrm{u}),$$

which in particular implies that $\mathscr{C}^\alpha(\mathsf{h}_\delta u, X) \in L^{4/3}(\mathfrak{m}) + L^2(\mathfrak{m})$. At this point we use the strong continuity of $\alpha \mapsto \mathsf{h}_\alpha$ at $\alpha = 0$ and the continuity of $\alpha \mapsto \boldsymbol{B}^\alpha$ in $(0,\infty)$ in order to conclude that $\mathscr{C}^\alpha(\mathsf{h}_\delta u, X) \to 0$ in $L^{4/3}(\mathfrak{m}) + L^2(\mathfrak{m})$ as $\alpha \downarrow 0$.

□

As in the Euclidean theory, we need some regularity on the vector field in order to prove the well-posedness of the continuity equation: in particular it turns out that an estimate of the commutator involves only the symmetric part of the derivative of the vector field (as proved in [24]). This structure can be recovered in our non-smooth setting: for this reason we introduce the following definition, which is the natural extension of Bakry's weak definition of Hessian from gradient vector fields to general vector fields (we refer to [19]).

Definition 2.2.8 (Vector fields with deformation in L^2)**.** *Let $X \in L^2(TM)$ be a vector field with $\mathrm{div}(X) \in L^2(\mathfrak{m}) + L^\infty(\mathfrak{m})$. We say that the deformation of X, $\mathrm{D}^{sym}X$, is in $L^2(\mathfrak{m})$ if there exists $c \ge 0$*

$$\left|\int \mathrm{D}^{sym}X(f,g)\,\mathrm{d}\mathfrak{m}\right| \le c\|\mathrm{D}f\|_{L^4(\mathfrak{m})}\|\mathrm{D}g\|_{L^4(\mathfrak{m})} \tag{2.2.3.6}$$

for all $f, g \in \mathbb{V}_4$ with $\Delta f, \Delta g \in L^4(\mathfrak{m})$, where

$$\int \mathrm{D}^{sym}X(f,g)\,\mathrm{d}\mathfrak{m} := -\frac{1}{2}\int \mathrm{d}f(X)\Delta g + \mathrm{d}g(X)\Delta f - \langle\nabla f, \nabla g\rangle \mathrm{div}X\,\mathrm{d}\mathfrak{m}. \tag{2.2.3.7}$$

In this case we denote by $\|\mathrm{D}^{sym}X\|_{L^2(\mathfrak{m})}$ the smallest constant c in (2.2.3.6).

Remark 2.2.9 (Relation between deformation and covariant derivative of a vector field)*.* We start observing that if (M, g) is a compact Riemannian manifold, Vol is its associated Riemannian volume and ∇X is the covariant derivative of a smooth vector field, then the expression

$$\langle\nabla g, \nabla\langle X, \nabla f\rangle\rangle + \langle\nabla f, \nabla\langle X, \nabla g\rangle\rangle - \langle X, \nabla\langle\nabla f, \nabla g\rangle\rangle = \langle\nabla_{\nabla f}X, \nabla g\rangle + \langle\nabla_{\nabla g}X, \nabla f\rangle$$

gives exactly twice the symmetric part of the tensor ∇X. Hence integrating over M and then integrating by parts, we obtain twice the expression in (2.2.3.7).

Let us see what happens in our non-smooth setting. Let $X \in W^{1,2}_C(TM)$ and recall the definition of covariant derivative ∇X given in (1.4.5.1). A direct computation gives:

$$\begin{aligned}
&\int h\ \nabla X : \big((\nabla f \otimes \nabla g) + (\nabla g \otimes \nabla f)\big)\,\mathrm{d}\mathfrak{m} \\
&\quad = -\frac{1}{2}\int \langle X, \nabla f\rangle \mathrm{div}(h\nabla g) + \langle X, \nabla g\rangle \mathrm{div}(h\nabla f) - \langle\nabla f, \nabla g\rangle \mathrm{div}(hX)\,\mathrm{d}\mathfrak{m}
\end{aligned} \tag{2.2.3.8}$$

for all $h, f, g \in \mathrm{Test}F(\mathrm{M})$. In particular for $h \equiv 1$ the right hand side of (2.2.3.8) gives exactly $2\int \mathrm{D}^{sym}X(f,g)\,\mathrm{d}\mathfrak{m}$. However this choice of a particular test function h makes $\mathrm{D}^{sym}X$ be a non local object.

However we point out that in order to define $\mathrm{D}^{sym}X$ we consider functions belonging to the closure of the space $V := \mathrm{span}\{\nabla f \otimes \nabla g : f, g \in \mathrm{Test}F(\mathrm{M})\} \subset L^2(T^{\otimes 2}\mathrm{M})$, while for the definition of covariant derivative the functions used are in the closure of the space $\tilde{V} := \mathrm{span}\{h\nabla f \otimes \nabla g : h, f, g \in \mathrm{Test}F(\mathrm{M})\}$, which is actually dense in $L^2(T^{\otimes 2}\mathrm{M})$. This in particular means that $\|\mathrm{D}^{sym}X\|_{L^2(T^{\otimes 2}\mathrm{M})} \leq \|\nabla X\|_{L^2(T^{\otimes 2}\mathrm{M})}$ and so when we have a bound on the norm $\|\nabla X\|_{L^2(T^{\otimes 2}\mathrm{M})}$, then the same bound holds also for the norm $\|\mathrm{D}^{sym}X\|_{L^2(T^{\otimes 2}\mathrm{M})}$. ■

We can now prove the following crucial lemma:

Lemma 2.2.10 (Commutator estimate)**.** *Let $Y \in W^{1,2}_C(T\mathrm{M})$ be such that $Y \in D(\mathrm{div})$. Then there exists a constant $c > 0$ such that*

$$\|\mathscr{C}^\varepsilon(Y,u)\|_{L^{4/3}} \leq c\,\|u\|_4\left[\|\nabla Y\|_2 + \|\mathrm{div}(Y)\|_2\right], \quad \forall \varepsilon \in (0,1),\ \forall u \in L^4(\mathfrak{m}). \tag{2.2.3.9}$$

proof In order to simplify the notation we set $g^\alpha := \mathsf{h}_\alpha g$. We observe that by duality inequality (2.2.3.9) is equivalent to the validity of

$$\begin{aligned}\int \mathscr{C}^\alpha(u,X)f\,\mathrm{d}\mathfrak{m} &= \int \mathrm{d}f^\alpha(X)u\,\mathrm{d}\mathfrak{m} - \int \mathrm{d}f(X)u^\alpha\,\mathrm{d}\mathfrak{m}\\ &\leq C\left[\|\mathrm{D}^{\mathrm{sym}}X\|_{4,2} + \|\mathrm{div}X\|_{L^2+L^\infty}\right]\|u\|_{L^4\cap L^2}\,\|f\|_{L^4\cap L^2(\mathfrak{m})},\end{aligned} \tag{2.2.3.10}$$

for every $f \in L^4 \cap L^2(\mathfrak{m})$; in particular, the density of $\mathscr{L}$ in $\mathbb{V}_4$ ensures that we can consider f of the form $f = \mathsf{h}_\varepsilon\varphi$ for some $\varphi \in \mathscr{L}$ and $\varepsilon > 0$. Moreover since both sides are continuous in u with respect to $L^4 \cap L^2(\mathfrak{m})$ convergence and $\mathscr{L}$ is dense in $L^4 \cap L^2(\mathfrak{m})$, it is also enough to establish it for $u = \mathsf{h}_\delta v$, for some $v \in \mathscr{L}, \delta > 0$.

Then if we define $F(\sigma) = \int \mathrm{d}f^\sigma(X)u^{\alpha-\sigma}\,\mathrm{d}\mathfrak{m}$, for $\sigma \in [0,\alpha]$, what we have to do is to find an estimate for

$$\int \mathrm{d}f^\alpha(X)u\,\mathrm{d}\mathfrak{m} - \int \mathrm{d}f(X)u^\alpha\,\mathrm{d}\mathfrak{m} = F(\alpha) - F(0).$$

The fact that $f = \mathsf{h}_\varepsilon\varphi$ with $\varphi \in \mathscr{L}$ and that (ii) in Proposition 2.2.7 ensure that the map $\sigma \mapsto \mathrm{d}f^\sigma(X) = \boldsymbol{B}^\varepsilon(\varphi^\sigma)$ is $C^1([0,\alpha], L^{4/3} + L^2(\mathfrak{m}))$ with

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}[\mathrm{d}f^\sigma(X)] = \boldsymbol{B}^\varepsilon(\Delta\varphi^\sigma).$$

We remark that (1.4.2.4) ensures that $\Delta u = \Delta\mathsf{h}_\delta v \in L^4 \cap L^2(\mathfrak{m})$ and so the map $\sigma \mapsto u^\sigma$ is $C^1([0,\alpha], L^4 \cap L^2(\mathfrak{m}))$ and by applying Leibniz rule we obtain

$$F(\alpha) - F(0) = \int_0^\alpha \left(\int \boldsymbol{B}^\varepsilon(\Delta\varphi^\sigma)u^{\alpha-\sigma} - \mathrm{d}f^\sigma(X)\Delta u^{\alpha-\sigma}\,\mathrm{d}\mathfrak{m}\right)\mathrm{d}\sigma.$$

Hence a direct application of (2.2.3.4) gives

$$\begin{aligned}\int \boldsymbol{B}^\varepsilon(\Delta\varphi^\sigma)u^{\alpha-\sigma}\,\mathrm{d}\mathfrak{m} &= \int \mathrm{div}((\mathsf{h}_\alpha\Delta\varphi^\sigma)X)u^{\alpha-\sigma} - (\mathsf{h}_\alpha\Delta\varphi^\sigma\mathrm{div}X)u^{\alpha-\sigma}\,\mathrm{d}\mathfrak{m}\\ &= -\int \Delta f^\sigma\mathrm{d}u^{\alpha-\sigma}(X) + (\mathrm{div}X)(\Delta f^\sigma)u^{\alpha-\sigma}\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

It is now convenient to estimate separately the two terms

$$I := -\int \Delta f^\sigma\mathrm{d}u^{\alpha-\sigma}(X) + \mathrm{d}f^\sigma(X)\Delta u^{\alpha-\sigma}\,\mathrm{d}\mathfrak{m}, \qquad II := -\int(\mathrm{div}X)(\Delta f^\sigma)u^{\alpha-\sigma}\,\mathrm{d}\mathfrak{m},$$

at fixed $\sigma \in (0,\alpha)$ and then integrate over σ.

First of all we add and subtract $\int (\mathrm{div} X)\mathrm{d}f^\sigma(\nabla u^{\alpha-\sigma})\,\mathrm{d}\mathfrak{m}$ to I and, recalling the definition

$$I = 2\int \mathrm{D}^{\mathrm{sym}}X(f^\sigma, u^{\alpha-\sigma})\,\mathrm{d}\mathfrak{m} - \int (\mathrm{div}X)\mathrm{d}f^\sigma(\nabla u^{\alpha-\sigma})\,\mathrm{d}\mathfrak{m}$$

By the assumptions made on $\mathrm{D}^{\mathrm{sym}}$, $\mathrm{div}X$ and the L^2-Γ and L^4-Γ inequalities to obtain that

$$|I| \le \left[2\left\|\mathrm{D}^{\mathrm{sym}}X\right\|_{4,4} + \left\|\mathrm{div}X\right\|_{L^2+L^\infty}\right] \frac{c}{\sqrt{\sigma(\alpha-\sigma)}} \left\|f\right\|_{L^4\cap L^2(\mathfrak{m})} \left\|u\right\|_{L^4\cap L^2(\mathfrak{m})}$$

where $c = 2c_4 + c_2$. The integration over $\sigma \in (0,\alpha)$ gives

$$\left|\int_0^\alpha I\,\mathrm{d}\sigma\right| \le \int_0^\alpha |I|\,\mathrm{d}\sigma \le c\pi\left[2\left\|\mathrm{D}^{\mathrm{sym}}X\right\|_{4,4} + \left\|\mathrm{div}X\right\|_{L^2+L^\infty}\right] \left\|f\right\|_{L^4\cap L^2(\mathfrak{m})} \left\|u\right\|_{L^4\cap L^2(\mathfrak{m})},$$

using the fact that

$$\int_0^\alpha \frac{\mathrm{d}\sigma}{\sqrt{\sigma(\alpha-\sigma)}} = \pi.$$

We turn then to the study of II: in order to do it we add and subtract

$$\int (\mathrm{div}X)(\Delta f^\sigma)u^\alpha\,\mathrm{d}\mathfrak{m} = \frac{\mathrm{d}}{\mathrm{d}\sigma}\int (\mathrm{div}X) f^\sigma u^\alpha\,\mathrm{d}\mathfrak{m},$$

hence we obtain

$$II = \int (\mathrm{div}X)(\Delta f^\sigma)(u^\alpha - u^{\alpha-\sigma})\,\mathrm{d}\mathfrak{m} - \frac{\mathrm{d}}{\mathrm{d}\sigma}\int (\mathrm{div}X) f^\sigma u^\alpha\,\mathrm{d}\mathfrak{m};$$

by Corollary 1.4.6 and the bound in (1.4.2.4) we can estimate the first part in II with

$$\begin{aligned}&\left|\int (\mathrm{div}X)(\Delta f^\sigma)(u^\alpha - u^{\alpha-\sigma})\,\mathrm{d}\mathfrak{m}\right| \\ &\qquad \le \left\|\mathrm{div}X\right\|_{L^2+L^\infty(\mathfrak{m})} \frac{c^\Delta}{\sigma}\min\left\{2, c^\Delta\log\left(1+\frac{\sigma}{\alpha-\sigma}\right)\right\} \left\|f\right\|_{L^4\cap L^2(\mathfrak{m})} \left\|u\right\|_{L^4\cap L^2(\mathfrak{m})}\end{aligned}$$

where $c^\Delta = 2c_4^\Delta + c_2^\Delta$. As for the second term in II it holds

$$\begin{aligned}-\int_0^\alpha \frac{\mathrm{d}}{\mathrm{d}\sigma}\int (\mathrm{div}X) f^\sigma u^\alpha\,\mathrm{d}\mathfrak{m}\,\mathrm{d}\sigma &= \int \mathrm{div}X(f - f^\alpha)u^\alpha\,\mathrm{d}\mathfrak{m}\,\mathrm{d}\sigma \\ &\le 2\left\|\mathrm{div}X\right\|_{L^2+L^\infty(\mathfrak{m})} \left\|f\right\|_{L^4\cap L^2(\mathfrak{m})} \left\|u\right\|_{L^4\cap L^2(\mathfrak{m})}.\end{aligned}$$

Then we observe that

$$\begin{aligned}\int_0^\alpha \min\left\{\frac{2}{\sigma}, \frac{c^\Delta}{\sigma}\log\left(1+\frac{\sigma}{\alpha-\sigma}\right)\right\}\mathrm{d}\sigma &\le \max\{2,c^\Delta\}\int_0^\alpha \min\left\{\frac{1}{\sigma},\frac{1}{\alpha-\sigma}\right\}\mathrm{d}\sigma \\ &= 2\log 2\max\{2,c^\Delta\}\end{aligned}$$

which proves the estimate in (2.2.3.10) and allows to conclude. □

A direct consequence of this result is the required convergence of the commutator in $L^1(\mathfrak{m})$:

Theorem 2.2.11. *Let $X \in W^{1,2}_C(T\mathrm{M})$ be such that $X \in D(\mathrm{div})$, then $\mathscr{C}^\varepsilon(u,X) \to 0$ in $L^1(\mathfrak{m})$ as $\varepsilon \downarrow 0$ for every $u \in L^4 \cap L^2(\mathfrak{m})$.*

proof For u of the form $u = \mathsf{h}_\delta v$ for some $v \in \mathcal{L}$ and $\delta > 0$, point (iv) in Proposition 2.2.7 guarantees that $\mathscr{C}^\alpha(u, X) \to 0$ in $L^{4/3} + L^2(\mathfrak{m})$ as $\alpha \downarrow 0$. Actually when $u \in \mathcal{L}$ directly from the definition of commutator in (2.2.1.10) follows that this convergence is also in $L^1(\mathfrak{m})$. Hence, the bound in (2.2.3.9), the density of $\mathscr{L}$ in $L^4 \cap L^2(\mathfrak{m})$, together with the fact that $u = \mathsf{h}_\delta v \in \mathscr{L}$, for every $v \in \mathscr{L}$ and every $\delta > 0$, allow to conclude that $\mathscr{C}^\alpha(u, X) \to 0$ in $L^1(\mathfrak{m})$ as $\alpha \downarrow 0$ also for every $u \in L^4 \cap L^2(\mathfrak{m})$. □

Therefore in our assumption in which the time dependent vector field is such that

$$|X| \in L^1_t(L^2_x), \qquad |\mathrm{div} X_t| \in L^1_t(L^\infty_x), \qquad |\nabla X_t| \in L^1_t(L^2_x),$$

just integrating the commutator estimate with respect to time, we have that

$$\int_0^T \|\mathscr{C}^\alpha(u_t, X_t)\| \, \mathrm{d}t \le C \|u\|_{L^\infty_t(L^4_x \cap L^2_x)} \left[\int_0^T \||\nabla X_t|\|_2 + \|\mathrm{div} X_t\|_2 \, \mathrm{d}t \right]$$

for all $u \in L^\infty_t(L^4_x \cap L^2_x)$ and $\alpha \in (0, \infty)$. Hence, by dominated convergence theorem, we get

$$\lim_{\alpha \downarrow 0} \int_0^T \|\mathscr{C}^\alpha(u_t, X_t)\|_1 \, \mathrm{d}t = 0. \tag{2.2.3.11}$$

2.3 Lagrangian point of view

Let us consider a countable subset $\mathscr{L}^*$ of the set $\mathscr{L}$ of all the Lipschitz functions with bounded support such that

$$\mathbb{Q}\mathscr{L}^* \text{ is dense in } W^{1,2}(\mathrm{M}) \text{ and a vector space over } \mathbb{Q} \tag{2.3.0.1}$$

and the extended distance in the sense of [8] (since it may take the value ∞) defined by

$$\mathrm{d}_{\mathscr{L}^*}(x, y) := \sup_{f \in \mathscr{L}^*} |f(x) - f(y)| \tag{2.3.0.2}$$

which has the property that

$$\lim_{m,n \to 0} \mathrm{d}_{\mathscr{L}^*}(x_n, x_m) \to 0 \quad \text{implies} \quad \lim_{n \to \infty} \mathrm{d}(x_n, x) \to 0 \quad \text{for some } x \in \mathrm{M}. \tag{2.3.0.3}$$

Lemma 2.3.1. *Let* $(\mathrm{M}, \mathsf{d}, \mathfrak{m})$ *be a* $\mathsf{RCD}(K, \infty)$ *space. Then there exists a countable dense set* $\mathscr{L}^* \subset \mathscr{L}$ *such that:*

1. *every* $f \in \mathscr{L}^*$ *is 1-Lipschitz;*
2. (2.3.0.1) *and* (2.3.0.3) *are satisfied;*
3. *the distance* $\mathrm{d}_{\mathscr{L}^*}$ *defined in* (2.3.0.2) *coincides with* d*;*
4. *for any vector field* $X \in L^2(T\mathrm{M})$ *it holds*

$$|X| = |X|_*, \ \mathfrak{m}\text{-}a.e. \text{ in } \mathrm{M}, \tag{2.3.0.4}$$

where $|X|_* := \sup\{|\mathrm{d}f(X)| : f \in \mathcal{L}^*\}$.

proof Let $(x_h) \subset \mathrm{M}$ be a dense set (the existence is guaranteed by the fact that (M, d) is separable) and set $f_{h,k} := (\mathrm{d}(x_h, \cdot) - k)^-$ for $h, k \in \mathbb{N}$. Obviously each $f_{h,k} \in \mathscr{L}$, being Lipschitz with bounded support. Now we recall that also $W^{1,2}(\mathrm{M})$ is separable and let us consider $(g_h) \subset \mathscr{L}$ dense in $W^{1,2}(\mathrm{M})$. Then we define

$$\mathscr{I} := \bigcup_{h,k=0}^{\infty} \{f_{h,k}\} \cup \bigcup_{h=0}^{\infty} \{g_h\} \quad \text{and} \quad \mathscr{L}^* := \{f \in \mathscr{I} : |Df| \leq 1, \mathfrak{m}\text{-a.e.}\}.$$

First of all we observe that the Sobolev-to-Lipschitz property ensures that every $f \in \mathscr{L}^*$ is 1-Lipschitz; moreover $\mathscr{L}^* \subset \mathscr{I}$ is such that $\mathbb{R}\mathscr{L}^* = \mathscr{I}$, and this guarantees the validity of (2.3.0.1).
At this point we note that each function $f_{h,k}$ is an element of $\mathscr{L}^*$ and this in particular means that $\mathscr{L}^*$ separates points. In turn this is a necessary condition to ensure that Cauchy sequences with respect to $\mathrm{d}_{\mathscr{L}^*}$ are Cauchy sequences with respect to d: we have then proved (2.3.0.3).

To show that the two distances coincide we first note that $\mathrm{d}_{\mathscr{L}^*} \leq \mathrm{d}$, while for the converse inequality we take $f = f_{h,k}$ in (2.3.0.2) with x_h arbitrarily close to x and k larger than $\mathrm{d}(x, y)$.

The only thing left to prove is then (2.3.0.4): the first step in this direction consists in proving that, possibly enlarging the set $\mathscr{L}^*$, for any $f \in \mathscr{L}^*$ the inequality $|\mathrm{d}f(X)| \leq |X|_*$ can be improved into $|\mathrm{d}f(X)| \leq |\mathrm{D}f||X|_*$, $\mathfrak{m}$-a.e. in M. The argument to prove it is based on a localization procedure, similar to the one in [10, Proposition 3.11]. Once we have this result, the conclusion follows by a density argument, where the curvature assumption plays a crucial role. We remark that the equality $\mathrm{d} = \mathrm{d}_{\mathscr{L}^*}$ and the property in (2.3.0.3) still holds for a further enlargement of $\mathscr{L}^*$, while we have to keep (2.3.0.1) true.

We now recall that for any $t > 0$, the operator h_t maps $L^2 \cap L^\infty(\mathfrak{m})$ into Test(M) and that the Bakry-Émery contraction estimate ensures that for any $f \in W^{1,2}(\mathrm{M})$ it holds

$$|\nabla \mathsf{h}_t f|^2 \leq e^{-2Kt} \mathsf{h}_t(|\nabla f|^2), \qquad \mathfrak{m}\text{-a.e.}, \forall t \geq 0.$$

Therefore, if $f \in \mathrm{Test}(\mathrm{M})$, then $|\nabla f| \in L^\infty(\mathfrak{m})$ and also $\mathsf{h}_t(|\nabla f|^2) \in \mathrm{Test}(\mathrm{M})$, which in particular means that there exists a Lipschitz representative $\bar{\mathsf{h}}_t(|\nabla f|^2)\colon \mathrm{M} \to \mathbb{R}$ such that $\mathsf{h}_t(|\nabla f|^2) = \bar{\mathsf{h}}_t(|\nabla f|^2)$, $\mathfrak{m}$-a.e., and $\mathrm{Lip}(\bar{\mathsf{h}}_t(|\nabla f|^2)) \leq \left\| |\nabla \mathsf{h}_t(|\nabla f|^2)| \right\|_{L^\infty(\mathfrak{m})}$.

Then we define the function $\zeta\colon \mathrm{M} \to [0, \infty)$ by $\zeta := e^{-Kt} \bar{\mathsf{h}}_t(|\nabla f|^2)^{1/2}$, which is in particular a bounded upper semicontinuous function such that $|\mathrm{D}f| \leq \zeta$, $\mathfrak{m}$-a.e. in M.

Now for any $\varepsilon > 0$, let $S_\varepsilon \in C^1(\mathbb{R})$ be a 1-Lipschitz truncation function, defined by $S_\varepsilon(r) := \varepsilon S_1(r/\varepsilon)$, where the 1-Lipschitz function $S_1(r)$ is equal to 1 in $B_1(0)$ while it is constantly zero outside $B_3(0)$; in particular $S_\varepsilon(r) = \varepsilon$ if $r \leq \varepsilon$.

We fix $f \in \mathrm{Test}(\mathrm{M})$ and for any $h \geq 1, \varepsilon > 0, M > 0$, such that $M \geq \sup_{B(x_h, 3\varepsilon)} \zeta$, we introduce a localization of f at $x_h \in \mathrm{M}$ given by

$$T_{h,\varepsilon,M}(f)(\cdot) := \frac{f(\cdot) - f(x_h)}{M} \wedge [S_\varepsilon \circ \mathrm{d}(\cdot, x_h)] \vee [-S_\varepsilon \circ \mathrm{d}(\cdot, x_h)].$$

We observe that $\mathrm{supp}(T_{h,\varepsilon,M}(f)) \subset B(x_h, 3\varepsilon)$ and that $T_{h,\varepsilon,M}(f)(x_h) = 0$. Moreover, directly from the definition, we have $T_{h,\varepsilon,M}(f) \in W^{1,2}(\mathrm{M})$ with $|\mathrm{D}T_{h,\varepsilon,M}(f)| \leq |\mathrm{D}f|/M \leq 1$ on $B(x_h, 3\varepsilon)$, while $|\mathrm{D}T_{h,\varepsilon,M}(f)| \equiv 0$ outside $B(x_h, 3\varepsilon)$. Hence the Sobolev-to-Lipschitz property guarantees that $T_{h,\varepsilon,M}(f)$ is a 1-Lipschitz function. These facts imply that $|T_{h,\varepsilon,M}(f)(x)| \leq \mathrm{d}(x_h, x)$ in M and so $T_{h,\varepsilon,M}(f)(x) = (f(x) - f(x_h))/M$ with $|T_{h,\varepsilon,M}(f)(x)| \leq \varepsilon$ in $B(x_h, \varepsilon)$.

In particular $T_{h,\varepsilon,M}(f)$ belongs to $\mathscr{L}$ and, thanks to the locality of the differential, it holds

$$\mathrm{d}f(X) = M \mathrm{d}T_{h,\varepsilon,M}(f)(X), \qquad \mathfrak{m}\text{-a.e. in } B(x_h, \varepsilon). \tag{2.3.0.5}$$

In the case in which $T_{h,\varepsilon,M}(f) \in \mathscr{L}^*$ for every $h \geq 1$ and any rational numbers $\varepsilon, M > 0$ such that $M \geq \sup_{B(x_h, 3\varepsilon)} \zeta$, (2.3.0.5) ensures that

$$|\mathrm{d}f(X)|(x) = M|\mathrm{d}T_{h,\varepsilon,M}(f)|(x) \leq M|X|_*(x), \qquad \mathfrak{m}\text{-a.e. in M}.$$

Then we pass to the infimum upon M, namely upon all the rational numbers greater than $\sup_{B(x_h,3\varepsilon)}\zeta$, and $h \geq 1$ and we let $\varepsilon \downarrow 0$; recalling that ζ is upper semicontinuous, we obtain

$$|\mathrm{d}f(X)|(x) \leq \limsup_{\varepsilon\downarrow 0} \inf_{h:\mathrm{d}(x_h,x)<\varepsilon} \sup_{B(x_h,3\varepsilon)} \zeta|X|_*(x) \leq \zeta(x)|X|_*(x), \quad \mathfrak{m}\text{-a.e. in M}.$$

We have obtained

$$|\mathrm{d}f(X)| \leq \zeta|X|_*, \qquad \mathfrak{m}\text{-a.e. in M}. \tag{2.3.0.6}$$

Since a priori we don't know if $T_{h,\varepsilon,M}(f)$ is in $\mathscr{L}^*$ for every h,ε and M as before, the next step consists in enlarging $\mathscr{L}^*$: we let $(f_n)_{n\geq 1}$ be a countable family of Lipschitz functions with bounded support such that the dilatations $(\lambda f_n)_{\lambda\in\mathbb{R},n\geq 1}$ forms a dense set in $W^{1,2}(\mathrm{M})$, whose existence is granted by the separability of $W^{1,2}(\mathrm{M})$ and the density of the Lipschitz functions in the Sobolev space. Hence we enlarge $\mathscr{L}^*$ with the union of all functions $T_{h,\varepsilon,M}(\mathsf{h}_t f_n)$ for $n,h \geq 1$ and rational numbers $t,\varepsilon,M > 0$ such that $M \geq \sup_{B(x_h,3\varepsilon)}\zeta$.

For every $n \geq 1$ and any rational $t > 0$, we consider (2.3.0.6) for $\mathsf{h}_t f_n$ and $\zeta := e^{-Kt}\bar{\mathsf{h}}_t(|\nabla f|^2)^{1/2}$, namely

$$|\mathrm{d}\mathsf{h}_t f_n(X)| \leq e^{-Kt}\bar{\mathsf{h}}_t(|\nabla f|^2)^{1/2}|X|_*, \quad \mathfrak{m}\text{-a.e. in M}.$$

Simply letting $t \downarrow 0$ we obtain

$$|\mathrm{d}f_n(X)| \leq |\mathrm{d}f_n||X|_*, \quad \mathfrak{m}\text{-a.e. in M}.$$

We use now the homogeneity to see that a similar inequality holds for λf_n in place of f_n for every $\lambda \in \mathbb{R}$. Then to conclude we take $g \in \mathcal{L}$ and $(g_k)_k \subseteq (\lambda f_n)_{\lambda\in\mathbb{R},n\geq 1}$ a sequence converging to g in $W^{1,2}(\mathrm{M})$. It holds

$$|\mathrm{d}g(X)| \leq \liminf_{k\to\infty} \Gamma(g_k)^{1/2}|X|_* + \Gamma(g_k - g)^{1/2}|X| = \Gamma(g)^{1/2}, \qquad \mathfrak{m}\text{-a.e. in M},$$

and we deduce that $|X| \leq |X|_*$, $\mathfrak{m}$-a.e. in M. $\square$

Definition 2.3.2 (ODE induced by a family of vector fields)**.** *Let $\boldsymbol{\eta} \in \mathscr{P}(C([0,T];\mathrm{M})$ and let $(X_t)_{t\in(0,T)}$ be a Borel family of vector fields. We say that $\boldsymbol{\eta}$ is concentrated on solutions to the ODE $\dot\eta = X_t(\eta)$ if*

$$f \circ \eta \in W^{1,1}(0,T) \ \text{and}\ \frac{\mathrm{d}}{\mathrm{d}t}(f\circ\eta)(t) = \mathrm{d}f(X_t)(\eta(t)), \ \text{for a.e. } t \in (0,T),$$

for $\boldsymbol{\eta}$-a.e. $\eta \in C([0,T],\mathrm{M})$, for all $f \in W^{1,2}(\mathrm{M})$.

Definition 2.3.3 (Regular Lagrangian flow)**.** *Let $(X_t) \in L^2([0,T],L^2_{\mathsf{loc}}(T\mathrm{M}))$. We say that $\mathrm{Fl}^{(X_t)} : [0,T]\times\mathrm{M}\to\mathrm{M}$ is a Regular Lagrangian Flow for (X_t) provided:*

i) The map $\mathrm{Fl}^{(X_t)}$ is Borel.

ii) There is $C > 0$ such that

$$(\mathrm{Fl}^{(X_t)}_s)_*\mathfrak{m} \leq C\mathfrak{m} \qquad \forall s \in [0,T]. \tag{2.3.0.7}$$

iii) For $\mathfrak{m}$-a.e. $x \in \mathrm{M}$ the curve $[0,T] \ni s \mapsto \mathrm{Fl}^{(X_t)}_s(x) \in \mathrm{M}$ is continuous and such that $\mathrm{Fl}^{(X_t)}_0(x) = x$.

iv) for every $f \in W^{1,2}(\mathrm{M})$ we have: for $\mathfrak{m}$-a.e. $x \in \mathrm{M}$ the function $s \mapsto f(\mathrm{Fl}^{(X_t)}_s(x))$ belongs to $W^{1,1}(0,T)$ and it holds

$$\frac{\mathrm{d}}{\mathrm{d}s} f(\mathrm{Fl}^{(X_t)}_s(x)) = \mathrm{d}f(X_s)(\mathrm{Fl}^{(X_t)}_s(x)) \qquad \mathfrak{m}\times\mathcal{L}^1|_{[0,T]}\text{-a.e.}(x,s). \tag{2.3.0.8}$$

Notice that it is due to property *(ii)* that property *(iv)* makes sense. Indeed, for given $X_s \in L^2(T\mathrm{M})$ and $f \in W^{1,2}(\mathrm{M})$ the function $\mathrm{d}f(X_s) \in L^1(\mathrm{M})$ is only defined $\mathfrak{m}$-a.e., so that (part of) the role of (2.3.0.7) is to grant that $\mathrm{d}f(X_s) \circ F_s$ is well defined $\mathfrak{m}$-a.e..

Lemma 2.3.4. *Let $\boldsymbol{\eta} \in \mathscr{P}(C([0,T]);\mathrm{M})$ be concentrated on solutions η to the ODE $\dot\eta = X_t(\eta)$, where $(X_t) \in L^1(L^2)$ and $\mu_t := (e_t)_\sharp \boldsymbol{\eta} \in \mathscr{P}(\mathrm{M})$ can be represented as $u_t\mathfrak{m}$, with $u \in L^\infty(L^2)$. Then the following two properties hold:*

1. *the family $(u_t)_{t\in(0,T)}$ is a weakly continuous solution to the continuity equation;*
2. *$\boldsymbol{\eta}$ is concentrated on $AC([0,T];(\mathrm{M},\mathrm{d}_{\mathscr{L}^*}))$ with*

$$|\dot\eta|(t) = |X|_*(\eta(t)) \qquad \text{for a.e. } t \in (0,T), \text{ for } \boldsymbol{\eta}\text{-a.e. } \eta. \tag{2.3.0.9}$$

proof Given $f \in \mathscr{L}^*$, for $\boldsymbol{\eta}$-a.e. η the map $t \mapsto f\circ\eta(t)$ is absolutely continuous with

$$f\circ\eta(t) - f\circ\eta(s) = \int_s^t \mathrm{d}f(X_r)(\eta(r))\,\mathrm{d}r, \quad \text{for all } s,t \in [0,T].$$

Now integrating with respect to $\boldsymbol{\eta}$ we obtain that the map $t \mapsto \int f u_t \,\mathrm{d}\mathfrak{m}$ is absolutely continuous in $[0,T]$ for all $f \in \mathscr{L}^*$ and a density argument ensures the weak continuity in duality with $\mathscr{L}$. In particular it holds $\mathrm{d}f(X_r)(\eta(t)) = (f\circ\eta)'(t)$ a.e. in $(0,T)$, for $\boldsymbol{\eta}$-a.e. η.

Then we use the fact that the marginals of $\boldsymbol{\eta}$ are absolutely continuous with respect to $\mathfrak{m}$ and Fubini's theorem, to get

$$\sup_{f\in\mathscr{L}^*} |(f\circ\eta)'(t)| = \sup_{f\in\mathscr{L}^*} |\mathrm{d}f(X_t)(\eta(t))| = |X_t|_*(\eta(t)), \text{ for a.e. } t\in(0,T), \text{ for } \boldsymbol{\eta}\text{-a.e. } \eta$$

and therefore

$$\mathrm{d}_{\mathscr{L}^*}(\eta(t),\eta(s)) = \sup_{f\in\mathscr{L}^*} |(f\circ\eta)(t) - (f\circ\eta)(s)| \le \int_s^t |X_t|_*(\eta(r))\,\mathrm{d}r, \text{ for all } s,t\in[0,T]$$

which proves that $\eta \in AC([0,T];(\mathrm{M},\mathrm{d}_{\mathscr{L}^*}))$ and that $|\dot\eta|(t) \le |X_t|_*(\eta(t))$, for a.e. $t\in(0,T)$. In order to conclude it suffices to observe that every $f \in \mathscr{L}^*$ is 1-Lipschitz with respect to $\mathrm{d}_{\mathscr{L}^*}$ and so for $\boldsymbol{\eta}$-a.e. η it holds

$$|X_t|_*(\eta(t)) = \sup_{f\in\mathscr{L}^*} |(f\circ\eta)'(t)| \le |\dot\eta|(t), \text{ for a.e. } t\in(0,T).$$

Hence we get (2.3.0.9) and we can conclude. □

We introduce now what it means for a curve in $\mathscr{P}(\mathrm{M})$ to be a solution of the continuity equation:

Definition 2.3.5 (Solutions of the continuity equation)**.** *Let $t \mapsto \mu_t \in \mathscr{P}(\mathrm{M})$ and $t \mapsto X_t \in L^0(T\mathrm{M})$, $t\in[0,T]$, be Borel maps. We say that they solve the continuity equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_t + \mathrm{div}(X_t\mu_t) = 0 \tag{2.3.0.10}$$

provided:

i) *$\mu_t \le C\mathfrak{m}$ for every $t\in[0,T]$ and some $C>0$,*

ii) *we have*

$$\int_0^T \int |X_t|^2 \,\mathrm{d}\mu_t\,\mathrm{d}t < \infty, \tag{2.3.0.11}$$

iii) *for any $f \in W^{1,2}(\mathrm{M})$ the map $t \mapsto \int f\,\mathrm{d}\mu_t$ is absolutely continuous and it holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int f\,\mathrm{d}\mu_t = \int \mathrm{d}f(X_t)\,\mathrm{d}\mu_t \qquad \text{a.e. } t.$$

2.4 The superposition principles

In this section we denote by $\mathbb{R}^\infty = \mathbb{R}^\mathbb{N}$, endowed with the product topology, and by $\pi^n := (p_1, \ldots, p_n)\colon \mathbb{R}^\infty \to \mathbb{R}^n$ the canonical projections from $\mathbb{R}^\infty$ to $\mathbb{R}^n$. On the space $\mathbb{R}^\infty$ we consider the distance defined by

$$\mathrm{d}_\infty(x, y) := \sum_{n=1}^{\infty} 2^{-n} \min\{1, |p_n(x) - p_n(y)|\},$$

which makes the space $(\mathbb{R}^\infty, \mathrm{d}_\infty)$ complete and separable. Respectively we consider the space $C([0, T]; \mathbb{R}^\infty)$ endowed with the distance

$$\delta(\eta, \tilde{\eta}) := \sum_{n=1}^{\infty} 2^{-n} \max_{t\in[0,T]} \min\{1, |p_n(\eta(t)) - p_n(\tilde{\eta}(t))|\}.$$

The space $(C([0, T]; \mathbb{R}^\infty), \delta)$ is complete and separable as well. Moreover we denote by $AC_w([0, T]; \mathbb{R}^\infty)$ the subspace of $C([0, T]; \mathbb{R}^\infty)$ consisting of all η such that $p_i \circ \eta \in AC([0, T]; \mathbb{R}^\infty)$ for all $i \geq 1$. We remark that for this class of curves the derivative $\eta' \in \mathbb{R}^\infty$ can still be defined a.e. in $(0, T)$, arguing componentwise.

We call regular cylindrical function any $f\colon \mathbb{R}^\infty \to \mathbb{R}$ that can be represented by

$$f(x) = \psi(\pi_n(x)) = \psi(p_1(x), \ldots, p_n(x)), \quad x \in \mathbb{R}^\infty$$

with $\psi\colon \mathbb{R}^n \to \mathbb{R}$ bounded and continuously differentiable with bounded derivative. Hence if f is regular cylindrical we can define the operator $\nabla f\colon \mathbb{R}^\infty \to c_0$, where c_0 is the space of sequences (x_n) null for n large enough, by

$$\nabla f(x) := \left(\frac{\partial \psi}{\partial z_1}(\pi_n(x)), \ldots, \frac{\partial \psi}{\partial z_n}(\pi_n(x)), 0, 0, \ldots \right).$$

Theorem 2.4.1 (Superposition principle in $\mathbb{R}^\infty$)**.** *Let $X\colon (0, T) \times \mathbb{R}^\infty \to \mathbb{R}^\infty$ be a Borel vector field. Let $\{\nu_t\}_{t\in(0,T)}$ be a weakly continuous (in duality with regular cylindrical functions) family of Borel probability measures such that*

$$\int_0^T \int |p_i(X_t)| \, \mathrm{d}\nu_t \, \mathrm{d}t < \infty, \quad \forall i \geq 1 \tag{2.4.0.1}$$

and, in the sense of distributions, for all f regular cylindrical it holds

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^\infty} f \, \mathrm{d}\nu_t = \int_{\mathbb{R}^\infty} \mathrm{d}f(X_t) \, \mathrm{d}\nu_t, \quad \textit{in } [0, T]. \tag{2.4.0.2}$$

Then there exists a Borel probability measure $\boldsymbol{\lambda}$ in $C([0, T]; \mathbb{R}^\infty)$ satisfying $(e_t)_\sharp \boldsymbol{\lambda} = \nu_t$ for all $t \in [0, T]$, concentrated on $\gamma \in AC_w([0, T]; \mathbb{R}^\infty)$ which are solution to the ODE $\dot{\gamma} = X_t(\gamma)$ a.e. in $(0, T)$.

Theorem 2.4.2 (Superposition principle in metric measure spaces)**.** *Let $(X_t)_{t\in(0,T)}$ be a Borel family of vector fields and let $\mu_t = u_t \mathfrak{m} \in \mathscr{P}(\mathrm{M})$, $0 \leq t \leq 1$, with $u \in L^\infty_t(L^2_x)$, be a solution of the continuity equation in the sense of Definition 2.3.5. Then there exists $\boldsymbol{\eta} \in \mathscr{P}(C([0, T]; (\mathrm{M}, \mathrm{d})))$ satisfying:*

(a) $\boldsymbol{\eta}$ is concentrated on solution η to the ODE $\dot{\eta} = X_t(\eta)$, in the sense of Definition 2.3.2;

(b) $\mu_t = (e_t)_\sharp \boldsymbol{\eta}$ *for any* $t \in [0,T]$.

proof We denote by f_i, $i \geq 1$, the elements of $\mathscr{L}^*$ and we define a continuous map $J\colon \mathrm{M} \to \mathbb{R}^\infty$ by

$$J(x) := (f_1(x), f_2(x), f_3(x), \dots)\,. \tag{2.4.0.3}$$

Note that property (2.3.0.3) ensures that this map is actually injective. Moreover if we take a sequence of compact sets $\{K_n\}_{n\in\mathbb{N}}$ such that $K_n \subset K_{n+1}$ and $\mathfrak{m}(\mathrm{M} \setminus K_n) \to 0$. Each one of the sets $J(K_n)$ is compact and so the set

$$J^* := \bigcup_{n=1}^{\infty} J(K_n) \subset J(\mathrm{M})$$

is σ-compact in $\mathbb{R}^\infty$.

Then we define a Borel probability measure $\nu_t \in \mathscr{P}(\mathbb{R}^\infty)$ by $\nu_t := J_\sharp \mu_t$ and a Borel vector field $Y\colon (0,T) \times \mathbb{R}^\infty \to \mathbb{R}^\infty$ by

$$Y_t^i := \begin{cases} (\mathrm{d}f_i(X_t)) \circ J^{-1}, & \text{on } J^*; \\ 0 & \text{otherwise.} \end{cases}$$

We point out that $f_i \in \mathscr{L}^*$ and so $|\mathrm{d}f_i| \leq 1$ ensures that

$$|Y_t^i| \circ J \leq |X_t|, \quad \mathfrak{m}\text{-a.e. in M}, \tag{2.4.0.4}$$

while the chain rule gives

$$\mathrm{d}\varphi(X_t)(x) = \sum_{i=1}^{n} \frac{\partial\psi}{\partial z_i}(f_1(x),\dots,f_n(x))\mathrm{d}f_i(X_t)(x) = \sum_{i=1}^{n} \frac{\partial\psi}{\partial z_i}(f_1(x),\dots,f_n(x))Y_t^i \circ J(x)$$

if $\varphi(x) = \psi(f_1(x),\dots,f_n(x))$, with $\psi\colon \mathbb{R}^n \to \mathbb{R}$ bounded and continuously differentiable with bounded derivative.

We observe that all the measures ν_t are concentrated on J^* and that (2.4.0.4), together with the fact that $\mu_t \ll \mathfrak{m}$ for every $t \in [0,T]$, ensures that $|Y_t^i| \leq |X_t| \circ J^{-1}$ ν_t-a.e. in $\mathbb{R}^\infty$. Therefore the hypotheses of Theorem 2.4.1 are satisfied by the family of measures ν_t and the vector field Y and so there exists a measure $\boldsymbol{\lambda} \in \mathscr{P}(C([0,T];\mathbb{R}^\infty))$ such that $(e_t)_\sharp\boldsymbol{\lambda} = \nu_t$ for all $t \in [0,T]$; moreover $\boldsymbol{\lambda}$ is concentrated on solutions $\gamma \in AC_w([0,T];\mathbb{R}^\infty)$ to the ODE $\dot\gamma = Y_t(\gamma)$ and, since all the measures ν_t are concentrated on J^*, it holds

$$\gamma(t) \in J^* \text{ for } \boldsymbol{\lambda}\text{-a.e. } \gamma, \text{ for all } t \in [0,T] \cap \mathbb{Q}.$$

We denote by N the $\boldsymbol{\lambda}$-negligible set for which the above property does not hold and we note that the curve $\eta := J^{-1} \circ \gamma\colon [0,T] \cap \mathbb{Q} \to \mathrm{M}$ is well defined for all $\gamma \in AC_w([0,T];\mathbb{R}^\infty) \setminus N$. Hence, for every $s,t \in [0,T] \cap \mathbb{Q}$ with $s < t$, we have

$$\begin{aligned} \sup_i |f_i(\eta(s)) - f_i(\eta(t))| = \sup_i |\gamma_i(s) - \gamma_i(t)| &\leq \sup_i \int_s^t |Y_r^i|(\gamma(r))\,\mathrm{d}r \\ &\leq \int_s^t |X_r|(J^{-1}(\gamma(r)))\chi_{J^*}(\gamma(r))\,\mathrm{d}r; \end{aligned}$$

the density of the points belonging to $[0,T] \cap \mathbb{Q}$ in $[0,T]$ and the above inequality show that $\gamma \in AC([0,T];\mathbb{R}^\infty)$, while the curve η is uniformly continuous in $[0,T] \cap \mathbb{Q}$ with respect to the distance $\mathrm{d}_{\mathscr{L}^*}$. However, the property in (2.3.0.3) and the fact that $\mathrm{d} = \mathrm{d}_{\mathscr{L}^*}$ guarantee that η has a unique extension to a continuous curve from $[0,T]$ to (M,d) and in particular that the image of $\gamma = J\circ\eta$ is actually contained in $J(\mathrm{M})$. Thus it make sense to define the $\boldsymbol{\lambda}$-measurable map $\Theta\colon \{\gamma\in AC([0,T];\mathbb{R}^\infty) : \gamma([0,T]\cap\mathbb{Q})\subset J^*\}\to C([0,T];(\mathrm{M},\mathrm{d}))$ which maps γ to $\Theta(\gamma) := J^{-1}\circ\gamma$ and the measure $\boldsymbol{\eta}\in\mathscr{P}(C([0,T];(\mathrm{M},\mathrm{d})))$ by

$$\boldsymbol{\eta} := \Theta_\sharp\boldsymbol{\lambda}$$

The definition of Θ, together with the fact that $(e_t)_\sharp\boldsymbol{\lambda} = \nu_t$ and that $(J^{-1})_\sharp\nu_t = \mu_t$, guarantee that $(e_t)_\sharp\boldsymbol{\eta} = \mu_t$.

Now we fix $i\geq 1$. Since $f_i\circ\eta = p_i\circ\gamma$ it follows that $f_i\circ\eta$ is absolutely continuous in $[0,T]$ and, from the definition of Y_i, for a.e. $t\in(0,T)$ and for $\boldsymbol{\eta}$-a.e. η it holds

$$(f_i\circ\eta)'(t) = (p_i\circ\gamma)'(t) = (\mathrm{d}f_i(Y_t))(J^{-1}(\gamma(t))) = (\mathrm{d}f_i(Y_t))(\eta(t)). \tag{2.4.0.5}$$

Thus in order to conclude the proof we have just to show that (2.4.0.5) extends from $\mathscr{L}^*$ to all of $\mathscr{L}$. The chain rule ensures that (2.4.0.5) holds for any truncation of f_i which is in $W^{1,2}(\mathrm{M})$, while the density of $\mathbb{Q}\mathscr{L}^*$ in $W^{1,2}(\mathrm{M})$ guarantees that for any $f\in\mathscr{L}$ we can find a sequence g_n satisfying:

(a) $g_n\to f$ in $W^{1,2}(\mathrm{M})$ and $\|g_n\|_\infty\leq\|f\|_\infty+1$;

(b) $g_n\circ\eta\in AC([0,T])$ and $(g_n\circ\eta)'(t) = \mathrm{d}g_n(X_t)(\eta(t))$ a.e. in $(0,T)$, for $\boldsymbol{\eta}$-a.e. η.

Now, since

$$\int\int_0^T |(f-g_n)(\eta(t))|\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\eta}(\eta) = \int_0^T\left(\int|f-g_n|\,\mathrm{d}\mu_t\right)\mathrm{d}t = \int_0^T\int|f-g_n|u_t\,\mathrm{d}\mathfrak{m}\,\mathrm{d}t\to 0, \tag{2.4.0.6}$$

possibly passing to a subsequence of (g_n), we can assume that $g_n\circ\eta\to f\circ\eta$ in $L^1(0,T)$ for $\boldsymbol{\eta}$-a.e. η. In order to achieve Sobolev regularity of $f\circ\eta$ we have to show the convergence of the derivatives of $g_n\circ\eta$, namely the convergence of $\mathrm{d}g_n(X_t)(\eta(t))$ to $\mathrm{d}f(X_t)(\eta(t))$. Again the same argument as in (2.4.0.6), the preservation of mass of the measures $\mu_t = u_t\mathfrak{m}$ proved in Proposition 2.2.1, the condition in (2.3.0.11) and the convergence of g_n to f in $W^{1,2}(\mathrm{M})$ give

$$\int\int_0^T|\mathrm{d}f(X_t)(\eta(t)) - \mathrm{d}g_n(X_t)(\eta(t))|\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\eta}(\eta) = \int_0^T\int|\mathrm{d}(f-g_n)(X_t)|u_t\,\mathrm{d}\mathfrak{m}\,\mathrm{d}t\to 0.$$

This means that, possibly passing to a further subsequence of (g_n), $\mathrm{d}g_n(X_t)(\eta(t))$ converges to $\mathrm{d}f(X_t)(\eta(t))$ in $L^1(0,T)$ for $\boldsymbol{\eta}$-a.e. η. □

2.5 Existence and uniqueness of regular Lagrangian flows

In this section we prove that, under some assumptions, there exists a unique regular Lagrangian flow Fl^X associated to the Borel family of vector fields $X = (X_t)_{t\in(0,T)}$ satisfying

$$X\in L^1_t(L^2(T\mathrm{M})). \tag{2.5.0.1}$$

Here uniqueness is understood in the pathwise sense: if $\mathrm{Fl}^{\mathrm{M}}_1$ and $\mathrm{Fl}^{\mathrm{M}}_2$ are two regular Lagrangian flows relative to M, then $\mathrm{Fl}^{\mathrm{M}}_1(\cdot,x) = \mathrm{Fl}^{\mathrm{M}}_2(\cdot,x)$ in $[0,T]$ for $\mathfrak{m}$-a.e. $x\in\mathrm{M}$.

The assumption we need is the following

Assumption 2.5.1. *The continuity equation has uniqueness of solutions in the class*

$$\mathscr{U}_+ := \{u\in L^\infty(L^1_+\cap L^\infty_+) : t\mapsto u_t \textit{ weakly continuous in } [0,T] \textit{ in duality with } \mathscr{L}\}$$

for any nonnegative initial datum $\bar{u} \in L^1 \cap L^\infty(\mathfrak{m})$*, and existence of solutions in the class*

$$\{u \in \mathscr{U}_+ : \|u_t\|_\infty \le C(X)\|\bar{u}\|_\infty, \quad \forall t \in [0,T]\},$$

for any nonnegative initial datum $\bar{u} \in L^1 \cap L^\infty(\mathfrak{m})$.

Theorem 2.5.2 (Existence and uniqueness of the regular Lagrangian flow)**.** *Let* $X = (X_t)_{t\in(0,T)}$ *be a Borel family of vector fields satisfying* (2.5.0.1) *and suppose that Assumption 2.5.1 holds true. Then there exists a unique regular Lagrangian flow relative to* X*.*

proof **Existence:** We start constructing a 'generalized' flow: we take $\bar{u} \equiv 1$ as initial datum and we first apply the assumption on the existence of a solution $u \in \mathcal{U}_+$ starting from $\bar{u}$ and with the property that $u_t \le C(X)$. Then the superposition principle in Theorem 2.4.2 yields to a measure $\boldsymbol{\eta} \in \mathscr{P}(C([0,T];\mathrm{M}))$ whose time marginals $u_t\mathfrak{m}$ are concentrated on the solutions of the ODE $\dot{\eta} = X_t(\eta)$.

Then Theorem 2.5.4 below (that we can apply since it uses just the uniqueness part of our assumptions relative to the continuity equation) gives us the representation

$$\boldsymbol{\eta} = \int_{\mathrm{M}} \delta_{\eta_x}\, \mathrm{d}\mathfrak{m}(x), \tag{2.5.0.2}$$

where the curve $\eta_x \in C([0,T];\mathrm{M})$ is such that $\eta_x(0) = x$ and $\dot{\eta}_x = X_t(\eta_x)$.

Now if we set $\mathrm{Fl}^{(X_t)}(\cdot,x) = \eta_x(\cdot)$, we have that $\mathrm{Fl}^{(X_t)} \colon [0,T]\times\mathrm{M} \to \mathrm{M}$ is a regular Lagrangian flow relative to the vector field X. As a matter of fact the map $\mathrm{Fl}^{(X_t)}$ is Borel in both the variables, being obtained by a disintegration of the plan $\boldsymbol{\eta} \in \mathscr{P}(C([0,T];\mathrm{M}))$ through the map e_0: indeed, if we denote by π_x the disintegration of $\boldsymbol{\eta}$ with respect to e_0, we have that for every $E \subset C([0,T];\mathrm{M})$ the map $x \mapsto \pi_x(E)$ is Borel. On the other hand, (2.5.0.2) ensures that $\pi_x = \delta_{\eta_x}$ for $\mathfrak{m}$-a.e. $x \in \mathrm{M}$: this in particular means that the set of the points where this identity does not hold is negligible, and so contained in a Borel negligible subset of M; moreover we recall the equality $\pi_x(E) = \eta^{-1}(\chi_E(x))$. Therefore, possibly redefining η in a Borel negligible set, the map $\eta \colon \mathrm{M} \to C([0,T];\mathrm{M})$ is Borel. Thus also the map

$$\begin{aligned} \tilde{\eta} \colon \mathrm{M}\times[0,T] &\to C([0,T];\mathrm{M})\times[0,T] \\ (x,t) &\mapsto (\eta_x(\cdot),t) \end{aligned}$$

is Borel and so it is its composition with the evaluation map e_t, since it is continuous. To conclude it suffices to observe that $\tilde{\eta}\circ e_t = \eta_x(\cdot) = \mathrm{Fl}^{(X_t)}(\cdot,x)$.

The only property left to verify is point (ii) in Definition 2.3.3, which is actually given by the following direct computation:

$$\mathrm{Fl}^{(X_t)}(t,\cdot)_\sharp(\bar{u}\mathfrak{m}) = (e_t)_\sharp\boldsymbol{\eta} = u_t\mathfrak{m} \le C(X)\mathfrak{m}.$$

Uniqueness: Let $\mathrm{Fl}^{(X_t)}$ and $\widetilde{\mathrm{Fl}}^{(X_t)}$ two regular Lagrangian flows and consider the measure

$$\boldsymbol{\eta} := \frac{1}{2}\int \delta_{\mathrm{Fl}^{(X_t)}(\cdot,x)} + \delta_{\widetilde{\mathrm{Fl}}^{(X_t)}(\cdot,x)}\,\mathrm{d}\mathfrak{m} \ \in\ \mathscr{P}(C([0,T];\mathrm{M})).$$

Again, Theorem 2.5.4 below applied to $\boldsymbol{\eta}$ ensures that $\mathrm{Fl}^{(X_t)}(\cdot,x) = \widetilde{\mathrm{Fl}}^{(X_t)}(\cdot,x)$ for $\mathfrak{m}$-a.e. $x \in \mathrm{M}$. □

Lemma 2.5.3. *Let* $\boldsymbol{\eta}_x$ *be a* $\mathfrak{m}$*-measurable family of positive finite measures in* $C([0,T];(\mathrm{M},\mathrm{d}))$ *satisfying the following property: for any* $t \in [0,T]$ *and for any pair of disjoint Borel sets* $E, E' \subset \mathrm{M}$ *it holds*

$$\boldsymbol{\eta}_x(\{\eta : \eta(t) \in E\}) \cdot \boldsymbol{\eta}_x(\{\eta : \eta(t) \in E'\}) = 0, \quad \mathfrak{m}\text{-}a.e.\ in\ \mathrm{M}. \tag{2.5.0.3}$$

Then $\boldsymbol{\eta}_x$ *is a Dirac mass for* $\mathfrak{m}$*-a.e.* $x \in \mathrm{M}$*.*

proof We start observing that $\boldsymbol{\eta}_x \in \mathscr{P}(C([0,T];\mathrm{M}))$ is a Dirac measure if and only if the pushforward measures $(e_t)_\sharp \boldsymbol{\eta}_x$ are Dirac measures. The only non trivial part to prove is the necessity one: if $(e_t)_\sharp \boldsymbol{\eta}_x = \delta_{x_t}$, then for every $t \in \mathbb{Q}$ the set $A_t := e_t^{-1}(\{x_t\}) = \{\eta \in C([0,T];\mathrm{M}) : \eta(t) = x_t\}$ is such that $\boldsymbol{\eta}_x(A_t) = 1$. Hence $A := \bigcap_{t\in\mathbb{Q}} A_t$ has also $\boldsymbol{\eta}_x(A) = 1$ (which in particular means that A is not empty). Now if η and $\tilde{\eta}$ are two elements in A, then $\eta(t) = x_t = \tilde{\eta}(t)$ for all $t \in \mathbb{Q}$ and so by continuity we conclude that $\eta = \tilde{\eta}$. Therefore A contains exactly one element η and we have $\boldsymbol{\eta} = \delta_\eta$.

Thus what we have to do is to prove that for a fixed $t \in (0,T]$ the measures $\lambda_x := (e_t)_\sharp \boldsymbol{\eta}_x \in \mathscr{P}(\mathrm{M})$ are Dirac for $\mathfrak{m}$-a.e.x.

Directly from (2.5.0.3) we have $\lambda_x(E)\lambda_x(E') = 0$ $\mathfrak{m}$-a.e. for any pair of disjoint Borel sets $E, E' \subset \mathrm{M}$. Let $\delta > 0$ and let us consider $\{R_j\}_{j\in\mathbb{N}}$ a partition of M into Borel sets each of them having a diameter less than δ. Then, since $\lambda_x(R_i)\lambda_x(R_j) = 0$ $\mathfrak{m}$-a.e. if $i \neq j$, we can find a decomposition of $\mathfrak{m}$-almost all of M into Borel sets A_j such that $\operatorname{supp}\lambda_x \subset \bar{R}_j$ for any $x \in A_j$: indeed it suffices to take $A_j = \{x : \lambda_x(R_j) > 0\} \setminus \cup_{i\neq j}\{x : \lambda_x(R_i) > 0\}$, which is non empty, thanks to (2.5.0.3). We can conclude using the arbitrariness of $\delta > 0$. □

Theorem 2.5.4 (No splitting criterion)**.** *Let $X = (X_t)_{t\in(0,T)}$ be a Borel family of vector fields satisfying* (2.5.0.1)*. Assume that the continuity equation induced by X has at most one solution in $\mathcal{U}_+$ for all $\bar{u} \in L^1 \cap L^\infty(\mathfrak{m})$. Let $\boldsymbol{\eta} \in \mathscr{P}(C([0,T];(\mathrm{M},\mathrm{d})))$ such that:*

(i) $\boldsymbol{\eta}$ is concentrated on solutions η to the ODE $\dot{\eta} = X_t(\eta)$;

(ii) there exists $L_0 \in [0,\infty)$ satisfying

$$(e_t)_\sharp \boldsymbol{\eta} \leq L_0 \mathfrak{m}, \quad \forall t \in [0,T].$$

Then the conditional measures $\boldsymbol{\eta}_x \in \mathscr{P}(C([0,T];(\mathrm{M},\mathrm{d})))$ induced by the map e_0 are Dirac masses for $(\mathrm{e}_0)_\sharp\boldsymbol{\eta}$-a.e. x. This in particular means that there exist $\eta_x \in C([0,T];(\mathrm{M},\mathrm{d}))$ which solve the ODE $\dot{\eta}_x = X_t(\eta_x)$ with the initial condition $\eta_x(0) = x$ and such that

$$\boldsymbol{\eta} = \int \delta_{\eta_x}\, \mathrm{d}(e_0)_\sharp \boldsymbol{\eta}(x). \tag{2.5.0.4}$$

proof We argue by contradiction supposing that $\boldsymbol{\eta}_x$ is not a Dirac measure in a set of $\bar{\mu} = \bar{u}\mathfrak{m}$ positive measure. Then Lemma 2.5.3 ensures the existence of two disjoint Borel sets $E, E' \subset \mathrm{M}$ and a Borel set $C \subset \mathrm{M}$ with $\mathfrak{m}(C) > 0$ such that

$$\boldsymbol{\eta}_x(\{\eta : \eta(t) \in E\}) \cdot \boldsymbol{\eta}_x(\{\eta : \eta(t) \in E'\}) > 0, \quad \forall x \in C.$$

Possibly passing to a smaller set C having still strictly positive $\mathfrak{m}$-measure we can assume that

$$0 < \boldsymbol{\eta}_x(\{\eta : \eta(t) \in E\}) \leq M\boldsymbol{\eta}_x(\{\eta : \eta(t) \in E'\}) \tag{2.5.0.5}$$

for some constant $M > 0$. Hence we define the two measures $\boldsymbol{\eta}^1$ and $\boldsymbol{\eta}^2$ whose disintegrations $\boldsymbol{\eta}^1_x$, $\boldsymbol{\eta}^2_x$ are given by

$$\boldsymbol{\eta}^1_x = \chi_C(x)\boldsymbol{\eta}_x \llcorner \{\eta : \eta(t) \in E\}, \qquad \boldsymbol{\eta}^2_x = M\chi_C(x)\boldsymbol{\eta}_x \llcorner \{\eta : \eta(t) \in E'\}$$

and we denote by μ^i_s, $s \in [0,t]$, the solution of the continuity equation induced by $\boldsymbol{\eta}^i$. Therefore it holds

$$\mu^1_0 = \boldsymbol{\eta}_x(\{\eta : \eta(t) \in E\})\mathfrak{m} \llcorner C \qquad \text{and} \qquad \mu^2_0 = M\boldsymbol{\eta}_x(\{\eta : \eta(t) \in E'\})\, \mathfrak{m} \llcorner C$$

and by (2.5.0.5) we have that $\mu_0^1 \leq \mu_0^2$. We observe that the fact that the two sets E, E' are disjoint ensures that μ_t^1 is orthogonal to μ_t^2. Indeed if we denote by $\boldsymbol{\eta}_{tx} \in \mathscr{P}(\mathrm{M})$ the pushforward measure of $\boldsymbol{\eta}_x$ through the map $\eta \mapsto \eta(t)$, then

$$\mu_t^1 = \int_C \boldsymbol{\eta}_{tx} \llcorner E \,\mathrm{d}\mu(x) \quad \perp \quad M \int_C \boldsymbol{\eta}_{tx} \llcorner E' \,\mathrm{d}\mu(x) = \mu_t^2.$$

In order to conclude we let $\rho\colon \mathrm{M} \to [0,T]$ be the density of μ_0^1 with respect to μ_0^2 and we define

$$\tilde{\boldsymbol{\eta}}_x^2 := M\rho(x)\chi_C(x)\boldsymbol{\eta}_x \llcorner \{\gamma : \gamma(t) \in E'\}.$$

We denote by $\tilde{\boldsymbol{\eta}}^2$ the measure whose disintegration is given by $\tilde{\boldsymbol{\eta}}_x^2$ and by $\tilde{\mu}_s^2$, $s \in [0,t]$, the solution of the continuity equation induced by $\tilde{\boldsymbol{\eta}}^2$.

At this point we remark that $\mu_s^i \leq \mu_s$ and so $\mu_s^i \in \mathcal{U}_+$; moreover since $\tilde{\boldsymbol{\eta}}^2 \leq \boldsymbol{\eta}^2$ we obtain that also $\tilde{\mu}_s^2 \in \mathcal{U}_+$. Moreover by construction we have $\mu_0^1 = \tilde{\mu}_0^2$, while μ_t^1 is orthogonal to μ_t^2, which in turn is a measure larger than $\tilde{\mu}_t^2$.

Thus we get the contradiction just observing that we have built two different solutions of the continuity equation with the same initial condition. □

Summing up the results in Theorem 2.2.2 and Theorem 2.2.6, which grant that Assumption 2.5.1 is verified, and recallig the estimate in (2.2.1.5), we have then proved the following result:

Theorem 2.5.5. *Let* $(X_t) \in L^2([0,T], W^{1,2}_{C,\mathsf{loc}}(T\mathrm{M})) \cap L^\infty([0,T], L^\infty(T\mathrm{M}))$ *be such that* $X_t \in D(\mathrm{div}_{\mathsf{loc}})$ *for a.e.* $t \in [0,T]$, *with*

$$\int_0^T \||\nabla X|\|_{L^2(\mathrm{M})} + \|\mathrm{div}(X_t)\|_{L^2(\mathrm{M})} + \|\big(\mathrm{div}(X_t)\big)^-\|_{L^\infty(\mathrm{M})} \,\mathrm{d}t < \infty. \tag{2.5.0.6}$$

Then a Regular Lagrangian Flow $F_s^{(X_t)}$ *for* (X_t) *exists and is unique, in the sense that if* $\tilde{F}^{(X_t)}$ *is another flow, then for* $\mathfrak{m}$*-a.e.* $x \in \mathrm{M}$ *it holds* $F_s(x) = \tilde{F}_s(x)$ *for every* $s \in [0,T]$. *Moreover it holds the quantitative bound*

$$(F_s^{(X_t)})_*\mathfrak{m} \leq \exp\Big(\int_0^s \|\big(\mathrm{div}(X_t)\big)^-\|_{L^\infty(\mathrm{M})} \,\mathrm{d}t\Big)\, \mathfrak{m} \qquad \forall s \in [0,T]. \tag{2.5.0.7}$$

Furthermore Theorem 2.5.4 together with Theorem 2.2.6 imply the following representation formula:

Theorem 2.5.6. *Let* (X_t) *be as in Theorem 2.5.5 and* $\bar{\mu} \in \mathscr{P}(\mathrm{M})$ *be such that* $\mu_0 \leq C\mathfrak{m}$ *for some* $C > 0$.

Then there exists a unique (μ_t) *such that* (μ_t, X_t) *solves the continuity equation* (2.3.0.10) *in the sense of Definition 2.3.5 and for which* $\mu_0 = \bar{\mu}$. *Moreover, such* (μ_t) *is given by*

$$\mu_s = (\mathrm{Fl}_s^{(X_t)})_*\bar{\mu} \qquad \forall s \in [0,T]. \tag{2.5.0.8}$$

In particular from Lemma 2.3.4 it follows that for $\mathfrak{m}$-a.e. x the curve $s \mapsto \mathrm{Fl}_s^{(X_t)}(x)$ is absolutely continuous with metric speed $|\dot{\mathrm{Fl}}_s^{(X_t)}(x)|$ given by

$$|\dot{\mathrm{Fl}}_s^{(X_t)}(x)| = |X_s|(\mathrm{Fl}_s^{(X_t)}(x)) \qquad a.e.\ s \in [0,T]. \tag{2.5.0.9}$$

It is worth to remark that in the case in which the family of vector fields is independent on the time, i.e., $X \equiv X_t$, then the Regular Lagrangian Flow is defined for any $t \geq 0$ and the uniqueness of the flow ensures that it satisfies the semigroup property

$$\mathrm{Fl}_t^{(X)} \circ \mathrm{Fl}_s^{(X)} = \mathrm{Fl}_{t+s}^{(X)} \quad \mathfrak{m} - a.e. \qquad \forall t, s \geq 0. \tag{2.5.0.10}$$

We conclude this section given an equivalent characterization of Regular Lagrangian flows in the case in which the family of vector fields (X_t) is such that

$$|X_t| \in L^\infty([0,T], L^\infty(\mathrm{M})). \tag{2.5.0.11}$$

We start observing that in this case a simple property valid for any $p \in [1,\infty)$ of Regular Lagrangian Flows is the following:

$$f_s \to f \quad \text{in } L^p(\mathrm{M}) \text{ as } s \to 0 \qquad \Rightarrow \qquad f_s \circ \mathrm{Fl}_s^{(X_t)} \to f \quad \text{in } L^p(\mathrm{M}) \text{ as } s \to 0. \tag{2.5.0.12}$$

This can be seen noticing that for any Lipschitz function $\tilde f$ with bounded support we have

$$\begin{aligned} \|f_s \circ \mathrm{Fl}_s^{(X_t)} - f\|_{L^p} &\leq \|f_s \circ \mathrm{Fl}_s^{(X_t)} - \tilde f \circ \mathrm{Fl}_s^{(X_t)}\|_{L^p} + \|\tilde t \circ \mathrm{Fl}_s^{(X_t)} - \tilde f\|_{L^p} + \|\tilde f - f\|_{L^p} \\ \text{by (2.3.0.7)} \qquad &\leq (C^{1/p}+1)\|f_t - \tilde f\|_{L^p} + \|\tilde f \circ \mathrm{Fl}_t^{(X_t)} - \tilde f\|_{L^p}. \end{aligned}$$

Since for $\mathfrak{m}$-a.e. $x \in \mathrm{M}$ the curve $s \mapsto \mathrm{Fl}_s^{(X_t)}(x)$ is Lipschitz and with metric speed bounded above by $\||X_t|\|_{L_t^\infty(L_x^\infty)}$, we have

$$|\tilde f(\mathrm{Fl}_t^X(x)) - \tilde f(x)| \leq |t| \operatorname{Lip}(\tilde f)\||X_t|\|_{L_t^\infty(L_x^\infty)} \qquad \mathfrak{m} - a.e.\ x \in \mathrm{M},$$

hence letting $t \to 0$ in the above we obtain

$$\varlimsup_{t\to 0} \|f_t \circ \mathrm{Fl}_t^X - f\|_{L^2} \leq (C^{1/p}+1)\|\tilde f - f\|_{L^p},$$

so that (2.5.0.12) follows from the arbitrariness of $\tilde f$ and the density of Lipschitz functions with bounded support in $L^p(\mathfrak{m})$.

Proposition 2.5.7. *Let $(X_t) \in L^2([0,T], L^2_{\mathsf{loc}}(T\mathrm{M}))$ be such that (2.5.0.11) holds and $F : [0,T] \times \mathrm{M} \to \mathrm{M}$ be a Borel map satisfying $(ii), (iii)$ of Definition 2.3.3. Then the following are equivalent:*

a) (iv) of Definition 2.3.3 holds, i.e. F is a Regular Lagrangian flow for (X_t).

b) for every $f \in W^{1,2}(\mathrm{M})$ the map $[0,T] \ni t \mapsto f \circ F_t \in L^2(\mathrm{M})$ is Lipschitz and for a.e. $t \in [0,T]$ it holds

$$\lim_{h\to 0} \frac{f \circ F_{t+h} - f \circ F_t}{h} = \mathrm{d}f(X_t) \circ F_t, \tag{2.5.0.13}$$

the limit being intended in $L^2(\mathrm{M})$.

c) for every $f \in W^{1,2}_{\mathsf{loc}}(\mathrm{M})$ the map $[0,T] \ni t \mapsto f \circ F_t \in L^2_{\mathsf{loc}}(\mathrm{M})$ is Lipschitz and (2.5.0.13) holds for a.e. $t \in [0,T]$ with the limit being intended in L^2_{loc}.

Moreover, if these holds and $X_t \equiv X$, then 'Lipschitz' in $(b), (c)$ can be replaced by 'C^1' and (2.5.0.13) holds for every $t \in [0,T]$.

proof
(a) $\Rightarrow$ **(b)** From (2.3.0.8) and Fubini's theorem we have that for $\mathfrak{m}$-a.e. x and $(\mathcal{L}^1|_{[0,T]})^2$-a.e. (s_0, s_1) it holds

$$f(\mathrm{Fl}_{s_T}^{(X_t)}(x)) - f(\mathrm{Fl}_{s_0}^{(X_t)}(x)) = \int_{s_0}^{s_T} \mathrm{d}f(X_s)(\mathrm{Fl}_s^{(X_t)}(x))\,\mathrm{d}s\,. \tag{2.5.0.14}$$

By the uniform bound (2.5.0.11) and (2.3.0.7) we deduce that $(\mathrm{d}f(X_\cdot) \circ \mathrm{Fl}_\cdot^{(X_t)}) \in L^\infty([0,T], L^2(\mathrm{M}))$, and thus the Bochner integral $\int_{s_0}^{s_1} \mathrm{d}f(X_s) \circ \mathrm{Fl}_s^{(X_t)}\,\mathrm{d}s$ is a well defined function in $L^2(\mathrm{M})$ which vary continuously in s_0, s_1. By (2.5.0.12) we also deduce that $s \mapsto f \circ \mathrm{Fl}_s^{(X_t)} \in L^2(\mathrm{M})$ is continuous, thus from (2.5.0.14) we obtain that

$$f \circ \mathrm{Fl}_{s_1}^{(X_t)} - f \circ \mathrm{Fl}_{s_0}^{(X_t)} = \int_{s_0}^{s_1} \mathrm{d}f(X_s) \circ \mathrm{Fl}_s^{(X_t)} \, \mathrm{d}s, \qquad \forall s_0, s_1 \in [0,T],\ s_0 < s_1,$$

where the identity is intended in $L^2(\mathrm{M})$ and the integral in the right hand side is the Bochner one. The Lipschitz continuity of $s \mapsto f \circ \mathrm{Fl}_s^{(X_t)} \in L^2(\mathrm{M})$ and (2.5.0.13) follow.

(**b**) $\Rightarrow$ (**c**) The assumption (2.5.0.11) together with (2.5.0.9) grant finite speed of propagation. Then the claim follows by a simple cut-off argument.

(**c**) $\Rightarrow$ (**a**) By assumption for every bounded set $B \subset \mathrm{M}$ we have

$$\chi_B\Big(f \circ \mathrm{Fl}_{s_1}^{(X_t)} - f \circ \mathrm{Fl}_{s_0}^{(X_t)}\Big) = \int_{s_0}^{s_1} \chi_B \mathrm{d}f(X_s) \circ \mathrm{Fl}_s^{(X_t)} \, \mathrm{d}s, \qquad \forall s_0, s_1 \in [0,T],\ s_0 < s_1$$

and thus from the arbitrariness of B and Fubini's theorem we conclude that for $\mathfrak{m}$-a.e. x it holds

$$f(\mathrm{Fl}_{s_1}^{(X_t)}(x)) - f(\mathrm{Fl}_{s_0}^{(X_t)}(x)) = \int_{s_0}^{s_1} \mathrm{d}f(X_s)(\mathrm{Fl}_s^{(X_t)}(x)) \, \mathrm{d}s, \qquad \mathcal{L}^2 - a.e.\ s_0, s_1 \in [0,T],\ s_0 < s_1.$$

Applying Lemma 2.1 of [7] we deduce that for $\mathfrak{m}$-a.e. x the function $t \mapsto f(\mathrm{Fl}_{s_1}^{(X_t)}(x))$ is in $W^{1,1}(0,T)$ and its distributional derivative is given by $\mathrm{d}f(X_s)(\mathrm{Fl}_s^{(X_t)}(x))$, thus concluding the proof.

C^1 **regularity:** It is sufficient to prove that $s \mapsto \mathrm{d}f(X) \circ \mathrm{Fl}_s^{(X)} \in L^2(\mathrm{M})$ (resp. L^2_{loc}) is continuous for $f \in W^{1,2}(\mathrm{M})$ (resp. $W^{1,2}_{\mathsf{loc}}(\mathrm{M})$). This is a direct consequence of (2.5.0.12) applied to the functions $f_s = \mathrm{d}f(X)$ (resp. $\chi_B \mathrm{d}f(X)$ for $B \subset \mathrm{M}$ Borel and bounded). $\square$

CHAPTER 3

Characterization of the flat torus among $\mathsf{RCD}^*(0,N)$-spaces

3.1 Overview of the chapter

In this chapter we prove Theorem 0.0.6, which is, as explained in the Introduction, a generalization of the second point in Bochner Theorem 0.0.3 to the setting of $\mathsf{RCD}^*(0,N)$ spaces. We recall that the correspondent of the first point of this result in this new framework has been obtained in Chapter 1, Proposition 1.4.42. Let us then describe the main outlines of the proof of this result and briefly illustrate the structure of this chapter.

Let $(\mathrm{M},\mathsf{d},\mathfrak{m})$ be a $\mathsf{RCD}^*(0,N)$ space such that $\dim(\mathcal{H}^1_{\mathrm{dR}}(\mathrm{M}))=N$. From the results obtained in Section 1.4.7 and the hypothesis on the dimension of the first cohomology group, we have the existence of N harmonic vector fields $X_1,\dots,X_N$ which are orthogonal in $L^2(T\mathrm{M})$, as we see in Section 3.4.1. Furthermore, since we are considering a space with non negative Ricci curvature, we have that these vector fields are Sobolev (i.e., belong to $H^{1,2}_C(T\mathrm{M})$), parallel, and divergence-free. In addition, all the X_i's are pointwise orthogonal and, up to normalization, we can assume that $|X_i|\equiv 1$ $\mathfrak{m}$-a.e. for every i. Hence, Theorem 2.5.5 grants that there exists a Regular Lagrangian flow for each one of these vector fields, which is defined for any $t\geq 0$, being every X_i independent on time.

It is worth also to notice that the fact that these vector fields are in $L^2(T\mathrm{M})$ forces the measure of the space to be finite.

Therefore, in order to prove Theorem 0.0.6, we study the map $\mathsf{T}\colon \mathrm{M}\times\mathbb{R}^N\to\mathrm{M}$, defined by

$$\mathsf{T}(x,\underline{a}=(a_1,\dots,a_N)):=\mathrm{Fl}^{X_1}_{a_1}\circ\cdots\circ\mathrm{Fl}^{X_N}_{a_N}(x).$$

In Section 3.3 we prove that the Regular Lagrangian flow $\mathrm{Fl}^{X_i}\colon\mathrm{M}\to\mathrm{M}$ associated to each one of the vector field X_i is an isometry, the argument being based on the crucial property that X_i is harmonic for every $i=1,\dots,N$. Moreover, we show that if Y is another harmonic vector field, then $\mathrm{Fl}^X_t\circ\mathrm{Fl}^Y_s=\mathrm{Fl}^{tX+sY}_1$ for any t,s. Hence we deduce that

$$\mathsf{T}(\mathsf{T}(x,\underline{a}),\underline{b})=\mathsf{T}(x,\underline{a}+\underline{b}),\qquad \forall x\in\mathrm{M},\ \underline{a},\underline{b}\in\mathbb{R}^N,$$

which allows to think at T as an action of $\mathbb{R}^N$ on M which is made of isometries, namely the map $\mathsf{T}(\cdot,\underline{a})\colon\mathrm{M}\to\mathrm{M}$ is an isometry for any $\underline{a}\in\mathbb{R}^N$.

The next step consists in showing that the action of $\mathbb{R}^N$ on M given by T is transitive (Proposition 3.4.6), namely we want to prove that

$$\text{for any couple of points } x,y\in\mathrm{M} \text{ there exists a vector } \underline{a}\in\mathbb{R}^N \text{ such that } \mathsf{T}(x,\underline{a})=y. \tag{3.1.0.1}$$

For this purpose, first of all we fix a point $y \in \mathrm{M}$ and we consider the two measures given by $\mu_0 = (\mathfrak{m}(B_R(y)))^{-1}\mathfrak{m}\llcorner B_R(y)$, for some $R > 0$, and $\mu_1 = \delta_y$; let (μ_t) be the unique W_2-geodesic connecting μ_0 and μ_1. Then we observe that for $\varepsilon \in (0, 1/2)$ the W_2-geodesic $t \mapsto \mu^\varepsilon_t := \mu_{\varepsilon+(1-2\varepsilon)t}$ satisfies the assumptions of Proposition 3.4.5: this means that we can find a family of vector fields $(v^\varepsilon_t) \in L^2(T\mathrm{M})$ for which $(\mu^\varepsilon_t, v^\varepsilon_t)$ solves the continuity equation, in the sense of (2.3.0.10). Moreover, for this family of vector fields (v^ε_t) there exists a Regular Lagrangian flow $(\mathrm{Fl}^{(v^\varepsilon_t)}_s)$. At this point we use the representation formula in Proposition 3.4.4 (which links the Regular Lagrangian flow of a family of vector fields on M with the action of the map T) and the continuity of T (which allows to pass to the limit as $\varepsilon \to 0$ in μ^ε_t) to conclude that every point $x \in B_R(y)$ is moved by T to the point y.

It is worth to underline that the proof of Proposition 3.4.4 is based on the study of Regular Lagrangian flows on the product space $\mathbb{R}^N \times \mathrm{M}$: for $(v_t) \in L^\infty([0, 1], L^2(T\mathrm{M})) \cap L^2([0, 1], W^{1,2}_C(T\mathrm{M}))$ with $(\mathrm{div}(v_t)) \in L^\infty([0, 1], L^\infty(\mathrm{M}))$, we consider the family of vector fields in $L^2(T(\mathbb{R}^N \times \mathrm{M}))$ which have the same behaviour of (v_t), but in the direction of $\mathbb{R}^N$ (this definition is made rigorous in (3.4.2.7)), and we look at their Regular Lagrangian flows. In particular, the study of these maps passes through the analysis of the (co)tangent module of the product space of two metric measure spaces $(\mathrm{M}_1, \mathsf{d}_1, \mathfrak{m}_1)$ and $(\mathrm{M}_2, \mathsf{d}_2, \mathfrak{m}_2)$: indeed, in Section 3.2, we see how we can relate the cotangent modules of M_1 and M_2 to the cotangent module of $\mathrm{M}_1 \times \mathrm{M}_2$, provided that these two metric measure spaces are such that the tensorization of Cheeger energy and the density in $W^{1,2}(\mathrm{M}_1 \times \mathrm{M}_2)$ of the algebra in Definition 3.2.3 hold.

Once we have proved (3.1.0.1), we fix a point $\bar{x} \in \mathrm{M}$ and denote by $\mathbb{G} \subset \mathbb{R}^N$ its stabilizer, namely the set of $\underline{a} \in \mathbb{R}^N$ such that $\mathsf{T}(\bar{x}, \underline{a}) = \bar{x}$. Again, the transitivity of T ensures that $\mathbb{G}$ does not depend on the particular choice of the point $\bar{x}$. Moreover we see that $\mathbb{G}$ is a subgroup of $\mathbb{R}^N$ which, by the continuity of T, is closed and discrete (Proposition 3.4.7).

Therefore, we equip the quotient space $\mathbb{R}^N/\mathbb{G}$ with the only Riemannian metric letting the quotient map be a Riemannian submersion. The distance induced by this metric is then

$$\mathsf{d}_{\mathbb{R}^N/\mathbb{G}}\big([\underline{a}], [\underline{b}]\big) = \min_{\substack{\underline{a}':[\underline{a}']=[\underline{a}] \\ \underline{b}':[\underline{b}']=[\underline{b}]}} |\underline{a}' - \underline{b}'|.$$

We observe that $\mathbb{R}^N/\mathbb{G}$ comes with the Haar measure, which coincides with the volume measure induced by the metric. In particular, the map T passes to the quotient and induces a map $\tilde{\mathsf{T}} : \mathbb{R}^N/\mathbb{G} \to \mathrm{M}$ via the formula:

$$\tilde{\mathsf{T}}([\underline{a}]) := \mathsf{T}(\bar{x}, \underline{a}).$$

In Theorem 3.4.8 we prove the main result of this chapter by showing first that the subgroup $\mathbb{G}$ of $\mathbb{R}^N$ is isomorphic to $\mathbb{Z}^N$, so that the quotient space $\mathbb{R}^N/\mathbb{G}$ is a flat torus $\mathbb{T}^N$, and then that the induced quotient map $\tilde{\mathsf{T}} : \mathbb{T}^N \to \mathrm{M}$ is an isometry verifying the property that $\tilde{\mathsf{T}}_*\mathfrak{m}_{\mathbb{T}^N} = c\mathfrak{m}$ for some $c > 0$.

3.2 Calculus on product spaces

3.2.1 Cotangent module and product of spaces

Let $(\mathrm{M}_1, \mathsf{d}_1, \mathfrak{m}_1)$ and $(\mathrm{M}_2, \mathsf{d}_2, \mathfrak{m}_2)$ be two metric measure spaces. Aim of this section is to relate the cotangent modules of $\mathrm{M}_1, \mathrm{M}_2$ to the cotangent module of the product space $\mathrm{M}_1 \times \mathrm{M}_2$, which will be always implicitly endowed with the product measure and the distance

$$(\mathsf{d}_1 \otimes \mathsf{d}_2)^2\big((x_1, x_2), (y_1, y_2)\big) := \mathsf{d}_1^2(x_1, y_1) + \mathsf{d}_2^2(x_2, y_2).$$

Let $\pi_i : \mathrm{M}_1 \times \mathrm{M}_2 \to \mathrm{M}_i$, $i = 1, 2$ be the canonical projections, observe that they are of local bounded deformation and recall from Proposition 1.3.13 and the discussion before it that $L^0(\mathrm{M}_2, L^0(T^*\mathrm{M}_1)) \sim [\pi_2^*]L^0(T^*\mathrm{M}_1)$ canonically carries the structure of $L^0(\mathrm{M}_1 \times \mathrm{M}_2)$-normed module.

First of all we prove the following useful result:

Proposition 3.2.1. *Let* $(\mathrm{M}_1, \mathsf{d}_1, \mathfrak{m}_1)$ *and* $(\mathrm{M}_2, \mathsf{d}_2, \mathfrak{m}_2)$ *be two metric measure spaces. Then there exists a unique* $L^0(\mathrm{M}_1 \times \mathrm{M}_2)$*-linear and continuous map*

$$\Phi_1 \colon L^0(\mathrm{M}_2, L^0(T^*\mathrm{M}_1)) \to L^0(T^*(\mathrm{M}_1 \times \mathrm{M}_2))$$

such that

$$\Phi_1(\widehat{\mathrm{d}g}) = \mathrm{d}(g \circ \pi_1) \qquad \forall g \in \mathcal{S}^2_{\mathsf{loc}}(\mathrm{M}_1), \tag{3.2.1.1}$$

where $\widehat{\mathrm{d}g} \colon \mathrm{M}_2 \to L^0(T^*\mathrm{M}_1)$ *is the function identically equal to* $\mathrm{d}g$*. Such map preserves the pointwise norm.*

In the same way, there is a unique $L^0(\mathrm{M}_1 \times \mathrm{M}_2)$*-linear and continuous map*

$$\Phi_2 \colon L^0(\mathrm{M}_1, L^0(T^*\mathrm{M}_2)) \to L^0(T^*(\mathrm{M}_1 \times \mathrm{M}_2))$$

such that

$$\Phi_2(\widehat{\mathrm{d}h}) = \mathrm{d}(h \circ \pi_1) \qquad \forall h \in \mathcal{S}^2_{\mathsf{loc}}(\mathrm{M}_2),$$

where $\widehat{\mathrm{d}h} \colon \mathrm{M}_1 \to L^0(T^*\mathrm{M}_2)$ *is the function identically equal to* $\mathrm{d}h$*, and such map preserves the pointwise norm.*

proof We shall prove the claims for Φ_1, as then the ones for Φ_2 follow by symmetry. The required L^0-linearity and (3.2.1.1) force the definition

$$\Phi_1(W) := \sum_{i,j} \chi_{A_i} \circ \pi_1 \chi_{B_j} \circ \pi_2 \, \mathrm{d}(g_{i,j} \circ \pi_1) \qquad \text{for} \qquad W = \sum_{i,j} \chi_{B_j}(\chi_{A_i} \mathrm{d}g_{i,j}) \tag{3.2.1.2}$$

where $(A_i), (B_j)$ are finite Borel partitions of $\mathrm{M}_1, \mathrm{M}_2$ respectively and $(g_{i,j}) \subset \mathcal{S}^2_{\mathsf{loc}}(\mathrm{M}_1)$. Since π_1 is 1-Lipschitz we have

$$|\Phi_1(W)| = \sum_i \chi_{B_i} \circ \pi_2 |\mathrm{d}(g_i \circ \pi_1)| \overset{(1.3.4.6)}{\leq} \sum_i \chi_{B_i} \circ \pi_2 |\mathrm{d}g_i| \circ \pi_1 = |W| \tag{3.2.1.3}$$

which shows both that (3.2.1.2) provides a good definition for $\Phi_1(W)$, in the sense that $\Phi_1(W)$ depends only on W and not on the way we write it as $\sum_{i,j} \chi_{B_j}(\chi_{A_i} \mathrm{d}g_{i,j})$, and that it is continuous. The definition also ensures that $\Phi_1(fW) = f\Phi_1(W)$ for $f \in L^0(\mathrm{M}_1 \times \mathrm{M}_2)$ of the form $\sum_{i,j} \alpha_{i,j} \chi_{A_i} \chi_{B_j}$ for $(A_i), (B_j)$ finite Borel partitions of $\mathrm{M}_1, \mathrm{M}_2$ respectively and $(\alpha_{i,j}) \subset \mathbb{R}$.

Since these functions are dense in $L^0(\mathrm{M}_1 \times \mathrm{M}_2)$ and the set of W's as in (3.2.1.2) is dense in $L^0(\mathrm{M}_1, L^0(T^*\mathrm{M}_2))$, this is enough to show existence and uniqueness of a L^0-linear and continuous Φ_1 for which (3.2.1.1) holds and, from (3.2.1.3), that for such Φ_1 we have

$$|\Phi_1(W)| \leq |W| \qquad \mathfrak{m}_1 \times \mathfrak{m}_2 - a.e.. \tag{3.2.1.4}$$

Thus to conclude it is sufficient to show that equality holds and, by the very same arguments just given, to this aim it is sufficient to show that

$$|\mathrm{d}g| \circ \pi_1 = |\mathrm{d}(g \circ \pi_1)| \quad \mathfrak{m}_1 \times \mathfrak{m}_2 - a.e. \qquad \forall g \in \mathcal{S}^2_{\mathsf{loc}}(\mathrm{M}_1). \tag{3.2.1.5}$$

It is now convenient to consider the map

$$\begin{array}{rcl} \mathrm{T}\colon C([0,1],\mathrm{M}_1)\times\mathrm{M}_2 & \to & C([0,1],\mathrm{M}_1\times\mathrm{M}_2) \\ (t\to\gamma_t,x_2) & \mapsto & t\to(\gamma_t,x_2). \end{array}$$

Notice that

$$\text{for any } x_2\in\mathrm{M}_2 \text{ the speed of } T(\gamma,x_2) \text{ is equal to the speed of } \gamma \text{ for a.e. } t \tag{3.2.1.6}$$

and fix $\mu\in\mathscr{P}(\mathrm{M}_2)$ such that $\mu\le\tilde{C}\mathfrak{m}_2$ for some $\tilde{C}>0$. Then for an arbitrary test plan $\boldsymbol{\pi}$ on M_1 define

$$\tilde{\boldsymbol{\pi}}:=\mathrm{T}_*(\boldsymbol{\pi}\times\mu)\in\mathscr{P}(C([0,1],\mathrm{M}_1\times\mathrm{M}_2))$$

and observe that $\tilde{\boldsymbol{\pi}}$ is a test plan on $\mathrm{M}_1\times\mathrm{M}_2$. Hence for any $g\in\mathcal{S}^2_{\mathsf{loc}}(\mathrm{M}_1)$ we have

$$\begin{aligned} \int|g\circ e_1-g\circ e_0|\,\mathrm{d}\boldsymbol{\pi} &= \int|g\circ\pi_1\circ e_1-g\circ\pi_1\circ e_0|\,\mathrm{d}\tilde{\boldsymbol{\pi}} \\ &\le\int_0^1\int|\mathrm{d}(g\circ\pi_1)|(\tilde{\gamma}_t)|\dot{\tilde{\gamma}}_t|\,\mathrm{d}\tilde{\boldsymbol{\pi}}(\tilde{\gamma})\,\mathrm{d}t \\ \text{by (3.2.1.6)}\qquad &= \int_0^1\int\left(\int|\mathrm{d}(g\circ\pi_1)|(\mathrm{T}(\gamma,x_2)_t)\,\mathrm{d}\mu(x_2)\right)|\dot{\gamma}_t|\,\mathrm{d}\boldsymbol{\pi}(\gamma)\,\mathrm{d}t, \end{aligned}$$

so that the arbitrariness of $\boldsymbol{\pi}$ gives that the function $\int|\mathrm{d}(g\circ\pi_1)|(\cdot,x_2)\mathrm{d}\mu(x_2)$ is a weak upper gradient of g. Therefore for $\mathfrak{m}_1$-a.e. x_1 we have

$$|\mathrm{d}g|(x_1)\le\int|\mathrm{d}(g\circ\pi_1)|(x_1,x_2)\,\mathrm{d}\mu(x_2)\overset{(1.3.4.6)}{\le}\int|\mathrm{d}g|\circ\pi_1(x_1,x_2)\,\mathrm{d}\mu(x_2)=|\mathrm{d}g|(x_1).$$

Hence the inequalities are equalities and the arbitrariness of μ gives (3.2.1.5). □

From now on we shall make two structural assumptions, which are needed since it seems hard to obtain any further relation between the cotangent modules in full generality.

Notation: In the following, for any function $f(x_1,x_2)$ on the product space $\mathrm{M}_1\times\mathrm{M}_2$, we define $f^{x_1}(\cdot):=f(x_1,\cdot)$ and, similarly, $f_{x_2}(\cdot):=f(\cdot,x_2)$.

Definition 3.2.2 (Tensorization of the Cheeger energy)**.** *We say that two metric measure spaces $(\mathrm{M}_1,\mathsf{d}_1,\mathfrak{m}_1)$ and $(\mathrm{M}_2,\mathsf{d}_2,\mathfrak{m}_2)$ have the property of tensorization of the Cheeger energy provided for any $f\in L^2(\mathrm{M}_1\times\mathrm{M}_2)$ the following holds: $f\in W^{1,2}(\mathrm{M}_1\times\mathrm{M}_2)$ if and only if*

- *$f^{x_1}\in W^{1,2}(\mathrm{M}_2)$ for $\mathfrak{m}_1$-a.e. $x_1\in\mathrm{M}_1$ and $\iint|\mathrm{d}f^{x_1}|^2\,\mathrm{d}\mathfrak{m}_2\,\mathrm{d}\mathfrak{m}_1(x_1)<\infty$*
- *$f_{x_2}\in W^{1,2}(\mathrm{M}_1)$ for $\mathfrak{m}_2$-a.e. $x_2\in\mathrm{M}_2$ with $\iint|\mathrm{d}f_{x_2}|^2\,\mathrm{d}\mathfrak{m}_1\,\mathrm{d}\mathfrak{m}_2(x_2)<\infty$*

and in this case it holds

$$|\mathrm{d}f|^2(x_1,x_2)=|\mathrm{d}f^{x_1}|^2(x_2)+|\mathrm{d}f_{x_2}|^2(x_1)\qquad\mathfrak{m}_1\times\mathfrak{m}_2\text{-a.e. }\ (x_1,x_2). \tag{3.2.1.7}$$

Notice that for $g\in L^\infty\cap W^{1,2}(\mathrm{M}_1)$ and $h\in L^\infty\cap W^{1,2}(\mathrm{M}_2)$ both with bounded support, the function $g\circ\pi_1\,h\circ\pi_2$ has bounded support and is in $L^\infty\cap W^{1,2}(\mathrm{M}_1\times\mathrm{M}_2)$, its differential being given by

$$\mathrm{d}(g\circ\pi_1\,h\circ\pi_2)=g\circ\pi_1\,\mathrm{d}(h\circ\pi_2)+h\circ\pi_2\,\mathrm{d}(g\circ\pi_1). \tag{3.2.1.8}$$

Definition 3.2.3 (Density of the product algebra)**.** *We say that two metric measure spaces* $(\mathrm{M}_1, \mathsf{d}_1, \mathfrak{m}_1)$ *and* $(\mathrm{M}_2, \mathsf{d}_2, \mathfrak{m}_2)$ *have the property of density of product algebra if the set*

$$\mathcal{A} := \Big\{ \sum_{j=1}^{n} g_j \circ \pi_1 \, h_j \circ \pi_2 \ : \ n \in \mathbb{N}, \quad \begin{array}{l} g_j \in L^\infty \cap W^{1,2}(\mathrm{M}_1) \text{ has bounded support} \\ h_j \in L^\infty \cap W^{1,2}(\mathrm{M}_2) \text{ has bounded support} \end{array} \ \forall j = 1, \ldots, n \Big\} \tag{3.2.1.9}$$

is dense in $W^{1,2}(\mathrm{M}_1 \times \mathrm{M}_2)$ *in the strong topology of* $W^{1,2}(\mathrm{M}_1 \times \mathrm{M}_2)$.

From now on we will always assume the following:

Assumption 3.2.4. $(\mathrm{M}_1, \mathsf{d}_1, \mathfrak{m}_1)$ *and* $(\mathrm{M}_2, \mathsf{d}_2, \mathfrak{m}_2)$ *are two metric measure spaces for which both the tensorization of Cheeger energy and the density of the product algebra hold.*

It is worth to underline that no couple of spaces $\mathrm{M}_1, \mathrm{M}_2$ are known for which Assumption 3.2.4 does not hold. On the other hand, it is unclear if that holds in full generality. The first results about the tensorization of Cheeger energies being given in [9] for the cases of two RCD spaces with finite mass, for our purposes the following result covers the cases of interest:

Proposition 3.2.5. *Let* $(\mathrm{M}_1, \mathsf{d}_1, \mathfrak{m}_1)$ *be a* $\mathsf{RCD}(K, \infty)$ *spaces and let* $(\mathrm{M}_2, \mathsf{d}_2, \mathfrak{m}_2)$ *be the Euclidean space* $\mathbb{R}^N$ *equipped with the Euclidean distance and the Lebesgue measure.*

Then both the tensorization of the Cheeger energy and the density of the product algebra hold.

proof In [39] it has been proved that for arbitrary $(\mathrm{M}_1, \mathsf{d}_1, \mathfrak{m}_1)$ and for $\mathrm{M}_2 = \mathbb{R}$ the tensorization of the Cheeger energy holds and the algebra $\mathcal{A}$ is dense in energy, i.e.: for any $f \in W^{1,2}(\mathrm{M}_1 \times \mathbb{R})$ there is $(f_n) \subset \mathcal{A}$ such that $f_n \to f$ and $|\mathrm{d} f_n| \to |\mathrm{d} f|$ in $L^2(\mathrm{M}_1 \times \mathbb{R})$.

If $(\mathrm{M}_1, \mathsf{d}_1, \mathfrak{m}_1)$ is infinitesimally Hilbertian (which is the case for RCD spaces), then the tensorization of the Cheeger energy ensures that $W^{1,2}(\mathrm{M}_1 \times \mathbb{R})$ is a Hilbert space, so that the uniform convexity of the norm grants that convergence in energy implies strong $W^{1,2}$-convergence.

Thus the thesis is true for $\mathrm{M}_2 = \mathbb{R}$. The general case follows by a simple induction argument. □

With this said, we shall now continue the investigation of the relation between cotangent modules and products of spaces, by introducing the following approximation result. It is worth to notice that the density of the product algebra is used to show that for $f \in \mathcal{S}^2_{\mathsf{loc}}(\mathrm{M}_1 \times \mathrm{M}_2)$ the map $x_2 \mapsto \mathrm{d} f_{x_2} \in L^0(T^*\mathrm{M}_1)$ is essentially separably valued.

Lemma 3.2.6. *Let* $(\mathrm{M}_1, \mathsf{d}_1, \mathfrak{m}_1)$ *and* $(\mathrm{M}_2, \mathsf{d}_2, \mathfrak{m}_2)$ *be two metric measure spaces satisfying Assumption 3.2.4.*

Then for every $f \in \mathcal{S}^2_{\mathsf{loc}}(\mathrm{M}_1 \times \mathrm{M}_2)$ *we have that* $f_{x_2} \in \mathcal{S}^2_{\mathsf{loc}}(\mathrm{M}_1)$ *for* $\mathfrak{m}_2$*-a.e.* x_2 *and the map* $x_2 \mapsto \mathrm{d} f_{x_2}$ *belongs to* $L^0(\mathrm{M}_2, L^0(T^*\mathrm{M}_1))$. *Moreover, for* $(f_n) \subset \mathcal{S}^2_{\mathsf{loc}}(\mathrm{M}_1 \times \mathrm{M}_2)$ *we have:*

$$\text{if} \qquad \mathrm{d} f_n \to \mathrm{d} f \ \text{ in } L^0(T^*(\mathrm{M}_1 \times \mathrm{M}_2)) \qquad \text{then} \qquad \mathrm{d}(f_n)_\cdot \to \mathrm{d} f_\cdot \ \text{ in } L^0(\mathrm{M}_2, L^0(T^*\mathrm{M}_1)). \tag{3.2.1.10}$$

Similarly for the roles of M_1 *and* M_2 *inverted. Finally, the identity* (3.2.1.7) *holds for any* $f \in \mathcal{S}^2_{\mathsf{loc}}(\mathrm{M}_1 \times \mathrm{M}_2)$.

proof Let $f = g \circ \pi_1 \, h \circ \pi_2$ for some $g \in L^\infty \cap W^{1,2}(\mathrm{M}_1)$ and $h \in L^\infty \cap W^{1,2}(\mathrm{M}_2)$ with bounded supports and notice that $\mathrm{d} f_{x_2} = h(x_2) \, \mathrm{d} g$ for every $x_2 \in \mathrm{M}_2$. Hence $x_2 \mapsto \mathrm{d} f_{x_2} \in L^2(\mathrm{M}_2, L^2(T^*\mathrm{M}_1))$.

By linearity, the same holds for a generic $f \in \mathcal{A}$. Now notice that for an arbitrary $f \in W^{1,2}(\mathrm{M}_1 \times \mathrm{M}_2)$, the identity (3.2.1.7) yields

$$|\mathrm{d} f_{x_2}|^2(x_1) \le |\mathrm{d} f|^2(x_1, x_2) \qquad \mathfrak{m}_1 \times \mathfrak{m}_2 - a.e. \ (x_1, x_2), \tag{3.2.1.11}$$

and thus

$$\| |\mathrm{d}f_\cdot - \mathrm{d}\tilde{f}_\cdot| \|_{L^2(\mathrm{M}_2, L^2(\mathrm{M}_1))} \le \|f - \tilde{f}\|_{W^{1,2}(\mathrm{M}_1 \times \mathrm{M}_2)}. \tag{3.2.1.12}$$

Hence for $f \in W^{1,2}(\mathrm{M}_1 \times \mathrm{M}_2)$ arbitrary, using the density of the product algebra we can find $(f_n) \subset \mathcal{A}$ $W^{1,2}$-converging to f, so that from (3.2.1.12) we see that $\mathrm{d}f_\cdot \in L^2(\mathrm{M}_2, L^2(T^*\mathrm{M}_1))$.

For general $f \in \mathcal{S}^2_{\mathsf{loc}}(\mathrm{M}_1 \times \mathrm{M}_2)$, find a sequence $(f_n) \subset W^{1,2}(\mathrm{M}_1 \times \mathrm{M}_2)$ as in (1.3.1.7) and use the locality of the differential to get that (3.2.1.11) holds even for $f \in \mathcal{S}^2_{\mathsf{loc}}(\mathrm{M}_1 \times \mathrm{M}_2)$. Thus, since clearly $\mathrm{d}f_n \to \mathrm{d}f$ in $L^0(T^*(\mathrm{M}_1 \times \mathrm{M}_2))$, from (3.2.1.11) we also get that $|\mathrm{d}(f_n)_\cdot - \mathrm{d}f_\cdot| \to 0$ in $L^0(\mathrm{M}_2, L^0(\mathrm{M}_1))$: this proves both that $\mathrm{d}f_\cdot \in L^0(\mathrm{M}_2, L^0(T^*\mathrm{M}_1))$ and that $\mathrm{d}(f_n)_\cdot \to \mathrm{d}f_\cdot$ in $L^0(\mathrm{M}_2, L^0(T^*\mathrm{M}_1))$.

Since this latter convergence does not depend on the particular choice of the sequence $(f_n) \subset \mathcal{S}^2_{\mathsf{loc}}(\mathrm{M}_1 \times \mathrm{M}_2)$ such that $\mathrm{d}f_n \to \mathrm{d}f$ in $L^0(T^*(\mathrm{M}_1 \times \mathrm{M}_2))$, we proved also (3.2.1.10).

The last claim follows along the same approximation argument using the continuity property (3.2.1.10) (and the analogous one with M_1 and M_2 inverted). □

At this point we take $\mathscr{M}_1, \mathscr{M}_2$, two L^0-normed modules on a space M. On the product $\mathscr{M}_1 \times \mathscr{M}_2$ we shall consider the structure of L^0-normed module given by: the product topology, the multiplication by L^0-functions given by $f(v_1, v_2) := (fv_1, fv_2)$ and the pointwise norm defined as

$$|(v_1, v_2)|^2 := |v_1|^2 + |v_2|^2.$$

Directly from the definitions it follows that these actually endow $\mathscr{M}_1 \times \mathscr{M}_2$ with the structure of L^0-normed module.

In particular, $L^0(\mathrm{M}_2, L^0(T^*\mathrm{M}_1)) \times L^0(\mathrm{M}_1, L^0(T^*\mathrm{M}_2))$ is a $L^0(\mathrm{M}_1 \times \mathrm{M}_2)$-normed module and we can define $\Phi_1 \oplus \Phi_2$ as

$$\begin{array}{rccc} \Phi_1 \oplus \Phi_2 \colon & L^0(\mathrm{M}_2, L^0(T^*\mathrm{M}_1)) \times L^0(\mathrm{M}_1, L^0(T^*\mathrm{M}_2)) & \to & L^0(T^*(\mathrm{M}_1 \times \mathrm{M}_2)) \\ & (\omega, \sigma) & \mapsto & \Phi_1(\omega) + \Phi_2(\sigma) \end{array}$$

We then have the following result:

Theorem 3.2.7. *Let* $(\mathrm{M}_1, \mathsf{d}_1, \mathfrak{m}_1)$ *and* $(\mathrm{M}_2, \mathsf{d}_2, \mathfrak{m}_2)$ *be two metric measure spaces such that Assumption 3.2.4 holds. Then* $\Phi_1 \oplus \Phi_2$ *is an isomorphism of modules, i.e. it is* $L^0(\mathrm{M}_1 \times \mathrm{M}_2)$*-linear, continuous, surjective and it satisfies*

$$|\Phi_1(\omega_\cdot) + \Phi_2(\sigma_\cdot)|^2 = |\omega_\cdot|^2 + |\sigma_\cdot|^2 \quad \mathfrak{m}_1 \times \mathfrak{m}_2 - a.e.. \tag{3.2.1.13}$$

for every $\omega_\cdot \in L^0(\mathrm{M}_2, L^0(T^*\mathrm{M}_1))$ *and* $\sigma_\cdot \in L^0(\mathrm{M}_1, L^0(T^*\mathrm{M}_2))$.

Moreover, for every $f \in \mathcal{S}^2_{\mathsf{loc}}(\mathrm{M}_1 \times \mathrm{M}_2)$ *it holds:*

$$\mathrm{d}f = \Phi_1(\mathrm{d}f_\cdot) + \Phi_2(\mathrm{d}f^\cdot). \tag{3.2.1.14}$$

proof From Proposition 3.2.1 it follows that $\Phi_1 \oplus \Phi_2$ is $L^0(\mathrm{M}_1 \times \mathrm{M}_2)$-linear and continuous. Taking into account that $L^0(\mathrm{M}_2, L^0(T^*\mathrm{M}_1))$ is generated by elements of the kind $\widehat{\mathrm{d}g}$ for $g \in \mathcal{S}^2(\mathrm{M}_1)$, where $\widehat{\mathrm{d}g} \in L^0(\mathrm{M}_2, L^0(T^*\mathrm{M}_1))$ is the function identically equal to $\mathrm{d}g$, and similarly for $L^0(\mathrm{M}_1, L^0(T^*\mathrm{M}_2))$, to prove (3.2.1.13) it is sufficient to show that

$$|\Phi_1(\widehat{\mathrm{d}g}) + \Phi_2(\widehat{\mathrm{d}h})|^2 = |\mathrm{d}g|^2 \circ \pi_1 + |\mathrm{d}h|^2 \circ \pi_2 \quad \mathfrak{m}_1 \times \mathfrak{m}_2 - a.e. \tag{3.2.1.15}$$

for any $g \in \mathcal{S}^2(\mathrm{M}_1)$, $h \in \mathcal{S}^2(\mathrm{M}_2)$. Fix such g, h and put $f := g \circ \pi_1 + h \circ \pi_2 \in \mathcal{S}^2_{\mathsf{loc}}(\mathrm{M}_1 \times \mathrm{M}_2)$. Notice that trivially $\mathrm{d}f_{x_2} = \mathrm{d}g$ and $\mathrm{d}f^{x_1} = \mathrm{d}h$ for any $x_1 \in \mathrm{M}_1$ and $x_2 \in \mathrm{M}_2$, hence from the tensorization of Cheeger energy (recall the last claim of Lemma 3.2.6 above) we have

$$|\Phi_1(\widehat{\mathrm{d}g}) + \Phi_2(\widehat{\mathrm{d}h})|^2 = |\mathrm{d}(g \circ \pi_1) + \mathrm{d}(h \circ \pi_2)|^2 = |\mathrm{d}f|^2 \overset{(3.2.1.7)}{=} |\mathrm{d}f_\cdot|^2 + |\mathrm{d}f^\cdot|^2 = |\mathrm{d}g|^2 \circ \pi_1 + |\mathrm{d}h|^2 \circ \pi_2$$

which is (3.2.1.15). Thus $\Phi_1 \oplus \Phi_2$ preserves the pointwise norm.

Now we prove (3.2.1.14). Let $g \in L^\infty \cap W^{1,2}(\mathrm{M}_1)$ and $h \in L^\infty \cap W^{1,2}(\mathrm{M}_2)$ be both with bounded support and consider $f := g \circ \pi_1\, h \circ \pi_2$. Then $f \in W^{1,2}(\mathrm{M}_1 \times \mathrm{M}_2)$ and the very definition of Φ_1, Φ_2 grant that

$$\mathrm{d}f = h \circ \pi_2\, \mathrm{d}(g \circ \pi_1) + g \circ \pi_1\, \mathrm{d}(h \circ \pi_2) = \Phi_1(h\, \mathrm{d}g) + \Phi_2(g\, \mathrm{d}h) = \Phi_1\big(\mathrm{d}f_{\cdot}\big) + \Phi_2\big(\mathrm{d}f^{\cdot}\big),$$

so that in this case (3.2.1.14) is proved. By linearity, we get that (3.2.1.14) holds for general $f \in \mathcal{A}$. Then using first the density of $\mathcal{A}$ in $W^{1,2}(\mathrm{M}_1 \times \mathrm{M}_2)$ and then property (1.3.1.7), taking into account the convergence property (3.2.1.10) we conclude that (3.2.1.14) holds for general $f \in \mathcal{S}^2_{\mathsf{loc}}(\mathrm{M}_1 \times \mathrm{M}_2)$, as claimed.

It remains to prove that $\Phi_1 \oplus \Phi_2$ is surjective. By (3.2.1.14) we know that its image contains the space of differential of functions in $\mathcal{S}^2_{\mathsf{loc}}(\mathrm{M}_1 \times \mathrm{M}_2)$, and thus L^0-linear combinations of them. Since it preserves the pointwise norm, its image must be closed and since $L^0(T^*(\mathrm{M}_1 \times \mathrm{M}_2))$ is generated by differentials of functions in $\mathcal{S}^2_{\mathsf{loc}}(\mathrm{M}_1 \times \mathrm{M}_2)$, this is sufficient to conclude. $\square$

3.2.2 Other differential operators in product spaces

In the previous section we have seen how the differential behaves under products of spaces. We shall now investigate other differentiation operators under the assumption that $\mathrm{M}_1, \mathrm{M}_2$ are infinitesimally Hilbertian.

We start with the following simple orthogonality statement:

Proposition 3.2.8. *Let* $(\mathrm{M}_1, \mathsf{d}_1, \mathfrak{m}_1)$ *and* $(\mathrm{M}_2, \mathsf{d}_2, \mathfrak{m}_2)$ *be infinitesimally Hilbertian spaces such that Assumption 3.2.4 holds. Then* $\mathrm{M}_1 \times \mathrm{M}_2$ *is also infinitesimally Hilbertian and for every* $\omega^1_{\cdot} \in L^0(\mathrm{M}_2, L^0(T^*\mathrm{M}_1))$ *and* $\omega^2_{\cdot} \in L^0(\mathrm{M}_1, L^0(T^*\mathrm{M}_2))$ *we have*

$$\big\langle \Phi_1(\omega^1_{\cdot}), \Phi_2(\omega^2_{\cdot}) \big\rangle = 0 \quad \mathfrak{m}_1 \times \mathfrak{m}_2 - a.e.. \tag{3.2.2.1}$$

proof The fact that $W^{1,2}(\mathrm{M}_1 \times \mathrm{M}_2)$ is Hilbert is a direct consequence of the tensorization of the Cheeger energy and the assumption that both $W^{1,2}(\mathrm{M}_1)$ and $W^{1,2}(\mathrm{M}_2)$ are Hilbert. For (3.2.2.1) notice that

$$|\omega^1_{\cdot}|^2 + |\omega^2_{\cdot}|^2 \overset{(3.2.1.13)}{=} |\Phi_1(\omega^1_{\cdot}) + \Phi_2(\omega^2_{\cdot})|^2 = |\Phi_1(\omega^1_{\cdot})|^2 + |\Phi_2(\omega^2_{\cdot})|^2 + 2\left\langle \Phi_1(\omega^1_{\cdot}), \Phi_2(\omega^2_{\cdot}) \right\rangle,$$

so that the conclusion follows recalling that Φ_1, Φ_2 preserve the pointwise norms. $\square$

By means of the musical isomorphisms (1.3.3.1) the map Φ_1 induces a map, still denoted Φ_1, from $L^0(\mathrm{M}_2, L^0(T\mathrm{M}_1))$ to $L^0(T(\mathrm{M}_1 \times \mathrm{M}_2))$ via:

$$\Phi_1(X_{\cdot}) := \Phi_1(X^\flat_{\cdot})^\sharp.$$

Similarly for Φ_2. It is clear that these newly defined Φ_1, Φ_2 have all the properties we previously proved for the same operators viewed as acting on forms. We also notice that for any $\omega_{\cdot} \in L^0(\mathrm{M}_2, L^0(T^*\mathrm{M}_1))$ and $X_{\cdot} \in L^0(\mathrm{M}_2, L^0(T\mathrm{M}_1))$ we have

$$\Phi_1(\omega_{\cdot})(\Phi_1(X_{\cdot}))(x_1, x_2) = \omega_{x_2}(X_{x_2})(x_1) \quad \mathfrak{m}_1 \times \mathfrak{m}_2 - a.e.\ (x_1, x_2). \tag{3.2.2.2}$$

Indeed, for $\omega_{\cdot} \equiv \mathrm{d}g$ and $X_{\cdot} \equiv \nabla \tilde{g}$ for $g, \tilde{g} \in \mathcal{S}^2_{\mathsf{loc}}(\mathrm{M}_1)$ this is a direct consequence of the definition of Φ_1 and the fact that Φ_1 preserves the pointwise norm (and hence the pointwise scalar product), then the general case follows by $L^0(\mathrm{M}_1 \times \mathrm{M}_2)$-bilinearity and continuity of both sides.

Proposition 3.2.9. *Let* $(\mathrm{M}_1, \mathsf{d}_1, \mathfrak{m}_1)$ *and* $(\mathrm{M}_2, \mathsf{d}_2, \mathfrak{m}_2)$ *be infinitesimally Hilbertian spaces such that Assumption 3.2.4 holds. Then* $X \in D(\mathrm{div}_{\mathsf{loc}}, \mathrm{M}_1)$ *if and only if* $\Phi_1(\hat{X}) \in D(\mathrm{div}_{\mathsf{loc}}, \mathrm{M}_1 \times \mathrm{M}_2)$, *where* $\hat{X} \in L^0(\mathrm{M}_2, L^0(T\mathrm{M}_1))$ *is the function identically equal to* X*, and in this case*

$$\mathrm{div}(\Phi_1(\hat{\mathrm{X}})) = \mathrm{div}(\mathrm{X}) \circ \pi_1.$$

proof From the very definition of divergence it is readily verified that the thesis is equivalent to

$$\int \mathrm{d}f(\Phi_1(\hat{X}))\,\mathrm{d}(\mathfrak{m}_1 \times \mathfrak{m}_2) = \iint \mathrm{d}f_{\cdot}(X)\,\mathrm{d}\mathfrak{m}_1\,\mathrm{d}\mathfrak{m}_2$$

for every $f \in W^{1,2}(\mathrm{M}_1 \times \mathrm{M}_2)$ with bounded support.

For such f we have

$$\begin{aligned}\int \mathrm{d}f(\Phi_1(\hat{X}))\,\mathrm{d}(\mathfrak{m}_1 \times \mathfrak{m}_2) &\overset{(3.2.1.14)}{=} \int \big(\Phi_1(\mathrm{d}f_{\cdot}) + \Phi_2(\mathrm{d}f^{\cdot})\big)(\Phi_1(\hat{X}))\,\mathrm{d}(\mathfrak{m}_1 \times \mathfrak{m}_2)\\ &\overset{(3.2.2.1)}{=} \int \big(\Phi_1(\mathrm{d}f_{\cdot})\big)(\Phi_1(\hat{X}))\,\mathrm{d}(\mathfrak{m}_1 \times \mathfrak{m}_2)\\ &\overset{(3.2.2.2)}{=} \iint \mathrm{d}f_{\cdot}(X)\,\mathrm{d}\mathfrak{m}_1\,\mathrm{d}\mathfrak{m}_2,\end{aligned}$$

hence the conclusion. □

A related property is the following:

Proposition 3.2.10. *Let* $(\mathrm{M}_1, \mathsf{d}_1, \mathfrak{m}_1)$ *and* $(\mathrm{M}_2, \mathsf{d}_2, \mathfrak{m}_2)$ *be infinitesimally Hilbertian spaces such that Assumption 3.2.4 holds. Let* $X = \Phi_1(X^1_{\cdot}) + \Phi_2(X^2_{\cdot}) \in L^2(T(\mathrm{M}_1 \times \mathrm{M}_2))$ *be such that:*

- $X^1_{x_2} \in D(\mathrm{div}, \mathrm{M}_1)$ *for* $\mathfrak{m}_2$*-a.e.* $x_2 \in \mathrm{M}_2$ *with* $\displaystyle\int \left|\mathrm{div}(X^1_{\cdot})\right|^2 \mathrm{d}(\mathfrak{m}_1 \times \mathfrak{m}_2) < \infty$
- $X^2_{x_1} \in D(\mathrm{div}, \mathrm{M}_2)$ *for* $\mathfrak{m}_1$*-a.e.* $x_1 \in \mathrm{M}_1$ *with* $\displaystyle\int \left|\mathrm{div}(X^2_{\cdot})\right|^2 \mathrm{d}(\mathfrak{m}_1 \times \mathfrak{m}_2) < \infty.$

Then $X \in D(\mathrm{div})$ *and*

$$\mathrm{div}(\mathrm{X})(x_1, x_2) = \mathrm{div}(\mathrm{X}^1_{x_2})(x_1) + \mathrm{div}(\mathrm{X}^2_{x_1})(x_2) \qquad \mathfrak{m}_1 \times \mathfrak{m}_2\text{-a.e. } (x_1, x_2). \tag{3.2.2.3}$$

proof For any $f \in W^{1,2}(\mathrm{M}_1 \times \mathrm{M}_2)$ with bounded support we have

$$\begin{aligned}\int \mathrm{d}f(X)\,\mathrm{d}(\mathfrak{m}_1 \times \mathfrak{m}_2) &\overset{(3.2.1.14)}{=} \int \big(\Phi_1(\mathrm{d}f_{\cdot}) + \Phi_2(\mathrm{d}f^{\cdot})\big)(\Phi_1(X^1_{\cdot}) + \Phi_2(X^2_{\cdot}))\,\mathrm{d}(\mathfrak{m}_1 \times \mathfrak{m}_2)\\ &\overset{(3.2.2.1)}{=} \int \Phi_1(\mathrm{d}f_{\cdot})\Phi_1(X^1_{\cdot}) + \Phi_2(\mathrm{d}f^{\cdot})\Phi_2(X^2_{\cdot})\,\mathrm{d}(\mathfrak{m}_1 \times \mathfrak{m}_2)\\ &\overset{(3.2.2.2)}{=} \int \Big(\int \mathrm{d}f_{\cdot}(X^1_{\cdot})\,\mathrm{d}\mathfrak{m}_1\Big)\mathrm{d}\mathfrak{m}_2 + \int \Big(\int \mathrm{d}f^{\cdot}(X^2_{\cdot})\,\mathrm{d}\mathfrak{m}_2\Big)\mathrm{d}\mathfrak{m}_2,\end{aligned}$$

which is the thesis. □

These last two statements produce analogous ones for the Laplacian:

Corollary 3.2.11. *Let* $(\mathrm{M}_1, \mathsf{d}_1, \mathfrak{m}_1)$ *and* $(\mathrm{M}_2, \mathsf{d}_2, \mathfrak{m}_2)$ *be infinitesimally Hilbertian spaces such that Assumption 3.2.4 holds. Then:*

i) $f \in D(\Delta_{\mathsf{loc}}, \mathrm{M}_1)$ *if and only if* $f \circ \pi_1 \in D(\Delta_{\mathsf{loc}}, \mathrm{M}_1 \times \mathrm{M}_2)$ *and in this case*

$$\Delta(f \circ \pi_1) = (\Delta f) \circ \pi_1.$$

ii) Let $f \in W^{1,2}(\mathrm{M}_1 \times \mathrm{M}_2)$ *be such that*

- *for* $\mathfrak{m}_1$*-a.e.* $x_1 \in \mathrm{M}_1$, $f^{x_1} \in D(\Delta, \mathrm{M}_2)$ *with* $\int \|\Delta f^{x_1}\|^2_{L^2(\mathrm{M}_2)}\, \mathrm{d}\mathfrak{m}_1 < \infty$,
- *for* $\mathfrak{m}_2$*-a.e.* $x_2 \in \mathrm{M}_2$, $f_{x_2} \in D(\Delta, \mathrm{M}_1)$ *with* $\int \|\Delta f_{x_2}\|^2_{L^2(\mathrm{M}_1)}\, \mathrm{d}\mathfrak{m}_2 < \infty$.

Then $f \in D(\Delta, \mathrm{M}_1 \times \mathrm{M}_2)$ *and*

$$\Delta f(x_1, x_2) = \Delta f_{x_2}(x_1) + \Delta f^{x_1}(x_2) \qquad \mathfrak{m}_1 \times \mathfrak{m}_2 - a.e.\ (x_1, x_2). \tag{3.2.2.4}$$

proof For the first claim simply notice that, directly from the definition, we have $f \in D(\Delta_{\mathsf{loc}}, \mathrm{M}_1)$ if and only if $\nabla f \in D(\mathrm{div}_{\mathsf{loc}}, \mathrm{M})$ and in this case $\mathrm{div}(\nabla f) = \Delta f$. Similarly for $f \circ \pi_1$. Then observe that (3.2.1.1) grants that $\nabla(f \circ \pi_1) = \Phi_1(\widehat{\nabla f})$ and apply Proposition 3.2.9 above to conclude.

The second claim follows by analogous considerations using Proposition 3.2.10 and the identity $\nabla f = \Phi_1(\nabla f_\cdot) + \Phi_2(\nabla f^\cdot)$ (recall (3.2.1.14)). □

3.2.3 Hessian on product spaces

In this section we continue the investigation of differential operators in product spaces by considering products of RCD spaces and the Hessian of those functions depending only on one variable. Recall from [10] (see also [9]) that the product of two $\mathsf{RCD}(K,\infty)$ spaces is $\mathsf{RCD}(K,\infty)$ and that the tensorization of the Cheeger energy in the sense of Definition 3.2.2 holds.

For the current purposes the following slightly stronger density property is necessary:

Definition 3.2.12 (Density of the product algebra - strong form)**.** *We say that two metric measure spaces* $(\mathrm{M}_1, \mathsf{d}_1, \mathfrak{m}_1)$ *and* $(\mathrm{M}_2, \mathsf{d}_2, \mathfrak{m}_2)$ *have the property of density of product algebra in the strong form if for* $\mathcal{A} \subset W^{1,2}(\mathrm{M}_1 \times \mathrm{M}_2)$ *defined as in* (3.2.1.9) *it holds: for* $f \in L^\infty \cap W^{1,2}(\mathrm{M}_1 \times \mathrm{M}_2)$ *there exists* $(f_n) \subset \mathcal{A}$ *uniformly bounded and* $W^{1,2}$*-converging to* f.

Remark 3.2.13. If M_1 is infinitesimally Hilbertian and M_2 the Euclidean space, such strong form of density holds. This is a consequence of the construction done in [39], which grants that for M_1 arbitrary and $\mathrm{M}_2 = \mathbb{R}$, for any $f \in L^\infty \cap W^{1,2}(\mathrm{M}_1 \times \mathrm{M}_2)$ we can find $(f_n) \subset \mathcal{A}$ uniformly bounded and such that $(f_n), (|\mathrm{d}f_n|)$ converge to $f, |\mathrm{d}f|$ in L^2 respectively. The infinitesimally Hilbertianity of M_1 and the tensorization of the Cheeger energy (proved in [39]) implies the infinitesimally Hilbertianity of $\mathrm{M}_1 \times \mathrm{M}_2$ and in turn this forces the $W^{1,2}$-convergence of the functions (f_n) above to f.

The case $\mathrm{M}_2 = \mathbb{R}^n$ then comes from an induction argument. ■

This extra density assumption is needed in the following approximation lemma in order to use the $L^\infty-$Lip regularization of the heat flow (see [9]). Such lemma is about approximation of test functions in the product with test functions depending on one variable only and in order to formulate the result it is convenient to introduce the algebra $\tilde{\mathcal{A}}$ as

$$\tilde{\mathcal{A}} := \Big\{ \sum_{j=1}^n g_{1,j} \circ \pi_1\, g_{2,j} \circ \pi_2 \ :\ n \in \mathbb{N}, \quad \begin{array}{l} g_{1,j} \in \mathrm{Test}(\mathrm{M}_1) \text{ has bounded support} \\ g_{2,j} \in \mathrm{Test}(\mathrm{M}_2) \text{ has bounded support.} \end{array} \Big\}$$

Notice that the calculus rules obtained in Section 3.2 ensure that $\tilde{\mathcal{A}} \subset \mathrm{Test}(\mathrm{M}_1 \times \mathrm{M}_2)$. We then have the following lemma about approximation of test functions with ones in $\tilde{\mathcal{A}}$; notice that a two-steps procedure is needed because the required uniform bound on the differentials prevents arguments by diagonalization.

Lemma 3.2.14. *Let* $(\mathrm{M}_1, \mathsf{d}_1, \mathfrak{m}_1)$ *and* $(\mathrm{M}_2, \mathsf{d}_2, \mathfrak{m}_2)$ *be two* $\mathsf{RCD}(K,\infty)$ *metric measure spaces for which the density of the product algebra holds in the strong form (Definition 3.2.12). Let* $f \in \mathrm{Test}(\mathrm{M}_1 \times \mathrm{M}_2)$ *be with bounded support and find* $\chi_1 \in \mathrm{Test}(\mathrm{M}_1)$, $\chi_2 \in \mathrm{Test}(\mathrm{M}_2)$ *with bounded support and such that* $\mathrm{supp}(f)$ *is contained in the interior of* $\{\chi_1 = 1\} \times \{\chi_2 = 1\}$ *(recall* (1.4.3.1)*) and for* $t > 0$ *put* $\tilde{f}_t := \chi_1 \circ \pi_1 \chi_2 \circ \pi_2 \, \mathsf{h}_t f$.

Then:

i) It holds

a) $\tilde{f}_t \to f$ *in* $W^{2,2}(\mathrm{M}_1 \times \mathrm{M}_2)$ *as* $t \downarrow 0$,

b) $\Delta \tilde{f}_t \to \Delta f$ *in* $L^2(\mathrm{M}_1 \times \mathrm{M}_2)$ *as* $t \downarrow 0$,

c) $\sup_{t\in(0,1)} \||\mathrm{d}\tilde{f}_t|\|_{L^\infty} < \infty$,

d) the sets $\mathrm{supp}(\tilde{f}_t)$ *are uniformly bounded for* $t \in (0,1)$,

ii) For every $t > 0$ *there exists a sequence* $(g_n) \subset \tilde{\mathcal{A}}$ *such that:*

a) $g_n \to \tilde{f}_t$ *in* $W^{2,2}(\mathrm{M}_1 \times \mathrm{M}_2)$ *as* $n \to \infty$,

b) $\Delta g_n \to \Delta \tilde{f}_t$ *in* $L^2(\mathrm{M}_1 \times \mathrm{M}_2)$ *as* $n \to \infty$,

c) $\sup_{n\in\mathbb{N}} \||\mathrm{d}\tilde{g}_n|\|_{L^\infty} < \infty$,

d) the sets $\mathrm{supp}(g_n)$ *are uniformly bounded,*

proof

(i) It is well known that $\mathsf{h}_t f \to f$ in $W^{1,2}(\mathrm{M}_1 \times \mathrm{M}_2)$ and $\Delta \mathsf{h}_t f \to \Delta f$ in $L^2(\mathrm{M}_1 \times \mathrm{M}_2)$ as $t \downarrow 0$. From the Leibniz rules for the gradient and the Laplacian and taking into account Proposition 3.2.1 and Corollary 3.2.11 we then see that $\tilde{f}_t \to f$ in $W^{1,2}(\mathrm{M}_1 \times \mathrm{M}_2)$ and $\Delta\tilde{f}_t \to \Delta f$ in $L^2(\mathrm{M}_1 \times \mathrm{M}_2)$ as $t \downarrow 0$. Convergence in $W^{2,2}$ then follows by (1.4.4.9). The uniform bounds on the supports is trivial by construction and the uniform bound on the differential follows by the Bakry-Émery estimate (see Theorem 7.2 in [5]) and the fact that $|\mathrm{d}f| \in L^\infty$.

(ii) Fix $t > 0$ and use (1.4.3.1) to find functions $\tilde{\chi}_1 \in \mathrm{Test}(\mathrm{M}_1)$, $\tilde{\chi}_2 \in \mathrm{Test}(\mathrm{M}_2)$ with bounded support such that $\mathrm{supp}(\tilde{f}_t)$ is contained in the interior of $\{\tilde{\chi}_1 = 1\} \times \{\tilde{\chi}_2 = 1\}$. Also, let $(f_n) \subset \mathcal{A}$ be uniformly bounded and $W^{1,2}$-converging to f and put

$$g_n := (\chi_1\tilde{\chi}_1) \circ \pi_1 \, (\chi_2\tilde{\chi}_2) \circ \pi_2 \, \mathsf{h}_t f_n \qquad \forall n \in \mathbb{N}.$$

We claim that the g_n's satisfy the thesis. Indeed, from the regularizing properties of the heat flow we know that $\mathsf{h}_t f_n \to \mathsf{h}_t f$ in $W^{1,2}(\mathrm{M}_1 \times \mathrm{M}_2)$ and $\Delta \mathsf{h}_t f_n \to \Delta \mathsf{h}_t f$ in $L^2(\mathrm{M}_1 \times \mathrm{M}_2)$. Since $(\chi_1\tilde{\chi}_1)\circ\pi_1 \, (\chi_2\tilde{\chi}_2)\circ\pi_2 \, \mathsf{h}_t f = \tilde{f}_t$, the same arguments used in the previous step grant that $g_n \to \tilde{f}_t$ in $W^{2,2}(\mathrm{M}_1 \times \mathrm{M}_2)$ and $\Delta g_n \to \Delta \tilde{f}_t$ in $L^2(\mathrm{M}_1 \times \mathrm{M}_2)$. The fact that the supports of the g_n's are uniformly bounded is obvious, and the uniform bound on the differentials follows from the uniform bounds on the f_n's and the $L^\infty - \mathrm{Lip}$ regularization property (see Theorem 7.3 in [5]).

Thus it remains to show that $g_n \in \tilde{\mathcal{A}}$ and since test functions form an algebra, to this aim it is sufficient to show that $\mathsf{h}_t f_n \in \tilde{\mathcal{A}}$. By the linearity of the heat flow, the fact that $\mathsf{h}_t h$ is a test function for $h \in L^\infty$ and $t > 0$ and performing if necessary a truncation argument on the various addends in $f_n \in \mathcal{A}$, to conclude it is sufficient to show that for $h_1 \in L^\infty(\mathrm{M}_1)$ and $h_2 \in L^\infty(\mathrm{M}_2)$ it holds

$$\mathsf{h}_t(h_1 \circ \pi_1 \, h_2 \circ \pi_2) = \mathsf{h}_t^{\mathrm{M}_1}(h_1) \circ \pi_1 \, \mathsf{h}_t^{\mathrm{M}_2}(h_2) \circ \pi_2 \qquad \forall t > 0, \tag{3.2.3.1}$$

where $\mathsf{h}_t^{\mathrm{M}_1}$, $\mathsf{h}_t^{\mathrm{M}_2}$ are the heat flows in $\mathrm{M}_1, \mathrm{M}_2$ respectively. To this aim notice that Corollary 3.2.11 grants that for h_1, h_2 in the domain of the Laplacian in the respective spaces it holds

$$\Delta(h_1 \circ \pi_1 \, h_2 \circ \pi_2) = h_1 \circ \pi_1 \, (\Delta h_2) \circ \pi_2 + h_2 \circ \pi_2 \, (\Delta h_1) \circ \pi_1$$

then observe that thanks to this fact the map sending $t \geq 0$ to the right hand side of (3.2.3.1), call it $\tilde{h}_t$, is absolutely continuous with values in $L^2(\mathrm{M}_1 \times \mathrm{M}_2)$ and its derivative is given by $\Delta \tilde{h}_t$. By the uniqueness of the heat flow we conclude that $\tilde{h}_t = \mathsf{h}_t(\tilde{h}_0)$, which is our claim. □

We then have the following result:

Proposition 3.2.15. *Let* $(\mathrm{M}_1, \mathsf{d}_1, \mathfrak{m}_1)$ *and* $(\mathrm{M}_2, \mathsf{d}_2, \mathfrak{m}_2)$ *be two* $\mathsf{RCD}(K, \infty)$ *spaces for which the density of the product algebra holds in the strong form (Definition 3.2.12) and let* $f \in W^{2,2}_{\mathsf{loc}}(\mathrm{M}_1)$*. Then* $f \circ \pi_1 \in W^{2,2}_{\mathsf{loc}}(\mathrm{M}_1 \times \mathrm{M}_2)$ *and*

$$\mathrm{Hess}(f \circ \pi_1)(\nabla g, \nabla \tilde{g})(x_1, x_2) = \mathrm{Hess}(f)(\nabla g_{x_2}, \nabla \tilde{g}_{x_2})(x_1), \qquad \mathfrak{m}_1 \times \mathfrak{m}_2 - a.e.\ (x_1, x_2) \tag{3.2.3.2}$$

for every $g, \tilde{g} \in \mathrm{Test}(\mathrm{M}_1 \times \mathrm{M}_2)$*.*

proof Directly from the definitions we see that the map sending $g, \tilde{g} \in \mathrm{Test}(\mathrm{M}_1 \times \mathrm{M}_2)$ to the right hand side of (3.2.3.2) defines an element of $L^2_{\mathsf{loc}}((T^*)^{\otimes 2}(\mathrm{M}_1 \times \mathrm{M}_2))$, hence to conclude it is sufficient to show that for such element the identity (1.4.4.1) holds.

Now consider the identity (1.4.4.1) defining the Hessian for functions in $W^{2,2}_{\mathsf{loc}}$ with $g := g_n$, where (g_n) is a sequence of test functions $W^{2,2}$-converging to some limit g, such that $\Delta g_n \to \Delta g$ in L^2 and with $\mathrm{supp}(g_n)$ and $\| |\mathrm{d}g_n| \|_{L^\infty}$ uniformly bounded: it is readily verified that in this case the two sides of (1.4.4.1) pass to the limit.

Thus by Lemma 3.2.14 above and the bilinearity and symmetry in $g, \tilde{g}$, to conclude it is sufficient to consider $g = \tilde{g}$ of the form $g = g_1 \circ \pi_1\, g_2 \circ \pi_2$ for $g_1 \in \mathrm{Test}(\mathrm{M}_1)$ and $g_2 \in \mathrm{Test}(\mathrm{M}_2)$ both with bounded support. For such g we have $g_{x_2} = g_2(x_2) g_1$, and thus $\nabla g_{x_2} = g(x_2) \nabla g_1$, so that our aim is to show that for any $h \in \mathrm{Test}(\mathrm{M}_1 \times \mathrm{M}_2)$ with bounded support it holds

$$\begin{aligned} - \int \langle \nabla (f \circ \pi_1), \nabla g \rangle \,\mathrm{div}(h \nabla g) + h \langle \nabla (f \circ \pi), \nabla \frac{|\nabla g|^2}{2} \rangle \,\mathrm{d}(\mathfrak{m}_1 \times \mathfrak{m}_2) \\ = \int h g_2^2 \circ \pi_2 \mathrm{Hess}(f)(\nabla g_1, \nabla g_1) \circ \pi_1 \,\mathrm{d}(\mathfrak{m}_1 \times \mathfrak{m}_2). \end{aligned} \tag{3.2.3.3}$$

Denoting for clarity $\mathrm{div}_1, \mathrm{div}_2$ the divergence operators in $\mathrm{M}_1, \mathrm{M}_2$ respectively and using formulas (3.2.1.8), (3.2.2.1) and (3.2.2.3), for $\mathfrak{m}_1 \times \mathfrak{m}_2$-a.e. (x_1, x_2) we have

$$\begin{aligned} &\big(\langle \nabla (f \circ \pi_1), \nabla g \rangle \,\mathrm{div}(h \nabla g) \big)(x_1, x_2) \\ &\quad = g_2(x_2) \langle \nabla f, \nabla g_1 \rangle (x_1) \big(g_2(x_2) \mathrm{div}_1(h_{x_2} \nabla g_1)(x_1) + g_1(x_1) \mathrm{div}_2(h^{x_1} \nabla g_2)(x_2) \big). \end{aligned} \tag{3.2.3.4}$$

From (3.2.1.7) we have

$$|\nabla g|^2 = g_2^2 \circ \pi_2 |\nabla g_1|^2 \circ \pi_1 + g_1^2 \circ \pi_1 |\nabla g_2|^2 \circ \pi_2$$

and thus recalling (3.2.2.1) we obtain

$$h \langle \nabla (f \circ \pi_1), \nabla \frac{|\nabla g|^2}{2} \rangle = h\, g_2^2 \circ \pi_2 \langle \nabla f, \nabla \frac{|\nabla g_1|^2}{2} \rangle \circ \pi_1 + h\, |\nabla g_2|^2 \circ \pi_2 \big(g_1 \langle \nabla f, \nabla g_1 \rangle \big) \circ \pi_1.$$

Adding up this identity and (3.2.3.4) and integrating, the conclusion (3.2.3.3) follows by the defining property (1.4.4.1) of $\mathrm{Hess}(f)$ and the trivial identity

$$\begin{aligned} &\int g_1(x_1) g_2(x_2) \langle \nabla f, \nabla g_1 \rangle (x_1) \mathrm{div}_2(h^{x_1} \nabla g_2)(x_2) \,\mathrm{d}\mathfrak{m}_1(x_1) \,\mathrm{d}\mathfrak{m}_2(x_2) \\ &\qquad = \int g_1(x_1) \langle \nabla f, \nabla g_1 \rangle (x_1) \int g_2 \mathrm{div}_2(h^{x_1} \nabla g_2) \,\mathrm{d}\mathfrak{m}_2 \,\mathrm{d}\mathfrak{m}(x_1) \\ &\qquad = - \int g_1(x_1) \langle \nabla f, \nabla g_1 \rangle (x_1) \int h^{x_1} |\nabla g_2|^2 \,\mathrm{d}\mathfrak{m}_2 \,\mathrm{d}\mathfrak{m}(x_1) \\ &\qquad = - \int h\, |\nabla g_2|^2 \circ \pi_2 \big(g_1 \langle \nabla f, \nabla g_1 \rangle \big) \circ \pi_1 \,\mathrm{d}(\mathfrak{m}_1 \times \mathfrak{m}_2) \end{aligned}$$

which shows that $f\circ\pi_1\in W^{2,2}_{\mathsf{loc}}(\mathrm{M}_1\times\mathrm{M}_2)$ and formula (3.2.3.2). □

3.3 Flow of harmonic vector fields on $\mathsf{RCD}(0,\infty)$ spaces

In this section we work on a fixed $\mathsf{RCD}(0,\infty)$ space $(\mathrm{M},\mathsf{d},\mathfrak{m})$ and study the Regular Lagrangian Flow of a fixed non-zero vector field $X\in L^2(T\mathrm{M})$ which is harmonic, i.e. $X^\flat\in D(\Delta_H)$ with $\Delta_H X^\flat=0$. Recalling (1.4.7.14) we have that $\mathrm{div}X=0$, while (1.4.8.5) grants that X is parallel, i.e. $X\in H^{1,2}_C(T\mathrm{M})$ with $\nabla X=0$. This latter property also implies that $|X|$ is constant (see [36] for the details about this last claim).

We can thus apply Theorem 2.5.5 to deduce that there exists and is unique the Regular Lagrangian Flow $(\mathrm{Fl}^{(X)}_t)$ of X. Aim of this section is to prove that:

i) the Fl^X_t's are measure preserving isometries

ii) if Y is another harmonic vector field, then $\mathrm{Fl}^X_t\circ\mathrm{Fl}^Y_s=\mathrm{Fl}^{tX+sY}_1$ for any t,s.

Notice that by analogy with the smooth case, one would expect to need only the conditions $\mathrm{div}X=0$, $\nabla X=0$ and that M is a $\mathsf{RCD}(K,\infty)$ space to get the above. Yet, it is unclear to us whether these are really sufficient, (part of) the problem being in the approximation procedure used in Proposition 3.3.3 which requires our stronger assumptions, namely the harmonicity of the vector field X.

In what comes next we shall occasionally use the following simple fact: for $T,S\colon\mathrm{M}\to\mathrm{M}$ Borel we have

$$T_*\mu=S_*\mu\quad\forall\mu\in\mathscr{P}(\mathrm{M})\text{ with bounded support and density}\qquad\Rightarrow\qquad T=S\quad\mathfrak{m}-a.e.. \tag{3.3.0.1}$$

Indeed, if $T\neq S$ on a set of positive measure, for some $r>0$ we would have $\mathsf{d}(T(x),S(x))>2r$ for a set of x's of positive measure and thus using the separability of M we would be able to find $\bar{x}$ such that $T_*\mathfrak{m}(B_r(\bar{x}))>0$. Thus $\mathfrak{m}(T^{-1}(B_r(\bar{x})))>0$ and letting $A\subset T^{-1}(B_r(\bar{x}))$ be any bounded Borel subset of positive $\mathfrak{m}$-measure, for $\mu:=\mathfrak{m}(A)^{-1}\mathfrak{m}|_A$ we would have that $T_*\mu$ and $S_*\mu$ are concentrated on disjoint sets, and thus in particular $T_*\mu\neq S_*\mu$.

With this said, we prove the following result, which shows that the flows of X and $-X$ are one the inverse of the other:

Lemma 3.3.1. *Let $(\mathrm{M},\mathsf{d},\mathfrak{m})$ be a $\mathsf{RCD}(0,\infty)$ space and X a harmonic vector field. Then for every $t\geq0$ the following identities hold $\mathfrak{m}$-a.e.:*

$$\mathrm{Fl}^{(-X)}_t\circ\mathrm{Fl}^{(X)}_t=\mathrm{Id}\qquad\textit{and}\qquad\mathrm{Fl}^{(X)}_t\circ\mathrm{Fl}^{(-X)}_t=\mathrm{Id}.$$

proof We shall prove the first identity for $t=1$, as then the rest follows by similar arguments. Let $\mu\in\mathscr{P}(\mathrm{M})$ be with bounded support and density, and consider the curves $[0,1]\ni t\mapsto\mu_t,\tilde\mu_t\in\mathscr{P}(\mathrm{M})$ defined as

$$\mu_t:=(\mathrm{Fl}^{(X)}_{1-t})_*\mu\qquad\text{and}\qquad\tilde\mu_t:=(\mathrm{Fl}^{(-X)}_t)_*(\mathrm{Fl}^{(X)}_1)_*\mu,$$

notice that $\mu_0=\tilde\mu_0$ and that they both solve the continuity equation (2.3.0.10) for $X_t=-X$ in the sense of Definition 2.3.5. By Theorem 2.5.6 we conclude that $\mu_1=\tilde\mu_1$, i.e.

$$\mu=(\mathrm{Fl}^{(-X)}_1\circ\mathrm{Fl}^{(X)}_1)_*\mu.$$

The conclusion follows by the arbitrariness of μ and (3.3.0.1). □

From this proposition and the semigroup property (2.5.0.10) of Regular Lagrangian Flows, it follows that defining $\mathrm{Fl}_{-t}^{(X)} := \mathrm{Fl}_t^{(-X)}$ for $t \geq 0$ we have

$$\mathrm{Fl}_t^{(X)} \circ \mathrm{Fl}_s^{(X)} = \mathrm{Fl}_{t+s}^{(X)} \qquad \mathfrak{m}\text{-a.e.} \qquad \forall\, t,s \in \mathbb{R}. \tag{3.3.0.2}$$

Proposition 3.3.2 (Preservation of the measure)**.** *Let* $(\mathrm{M},\mathsf{d},\mathfrak{m})$ *be a* $\mathsf{RCD}(0,\infty)$ *space and* X *a harmonic vector field. Then for every* $t \in \mathbb{R}$ *we have*

$$(\mathrm{Fl}_t^{(X)})_*\mathfrak{m} = \mathfrak{m}. \tag{3.3.0.3}$$

proof Simply notice that from $\mathrm{div}(X) = \mathrm{div}(-X) = 0$ and (2.5.0.7), for any $t \geq 0$ we have

$$\mathfrak{m} = (\mathrm{Fl}_t^{(X)} \circ \mathrm{Fl}_{-t}^{(X)})_*\mathfrak{m} = (\mathrm{Fl}_t^{(X)})_*(\mathrm{Fl}_t^{(-X)})_*\mathfrak{m} \leq (\mathrm{Fl}_t^{(X)})_*\mathfrak{m} \leq \mathfrak{m}$$

forcing the inequalities to be equalities. □

We can now prove the following lemma, which is key to show that $\mathrm{Fl}_t^{(X)}$ is an isometry.

Proposition 3.3.3 (Euler's equation for X)**.** *Let* $(\mathrm{M},\mathsf{d},\mathfrak{m})$ *be a* $\mathsf{RCD}(0,\infty)$ *space and* X *a harmonic vector field. Then for any* $f \in W^{1,2}(\mathrm{M})$ *it holds*

$$\mathsf{h}_t\left(\big\langle \nabla f, X\big\rangle\right) = \big\langle \nabla \mathsf{h}_t f, X\big\rangle, \quad \mathfrak{m}\text{-}a.e., \forall t \geq 0. \tag{3.3.0.4}$$

Moreover, for every $f \in D(\Delta)$ *with* $\Delta f \in W^{1,2}(\mathrm{M})$*, we have* $\langle \nabla f, X\rangle \in D(\Delta)$ *and*

$$\Delta\big\langle \nabla f, X\big\rangle = \big\langle \nabla \Delta f, X\big\rangle, \quad \mathfrak{m}\text{-}a.e.. \tag{3.3.0.5}$$

proof We apply (1.4.8.13) in our space to the form $X^\flat + \varepsilon \mathrm{d} f$ to obtain

$$|\mathsf{h}_{H,t}(X^\flat + \varepsilon\, \mathrm{d} f)|^2 \leq \mathsf{h}_t(|X + \varepsilon \nabla f|^2). \tag{3.3.0.6}$$

We have already observed that the fact that $X^\flat$ is harmonic grants that $|X|$ is constant, say $|X| \equiv c$. The harmonicity also grants that $\mathsf{h}_{H,t}(X^\flat) = X^\flat$ for every $t \geq 0$, hence we have $|\mathsf{h}_{H,t}(X^\flat)|^2 \equiv c^2 \equiv \mathsf{h}_t(|X|^2)$ for any $t \geq 0$. Therefore,

$$c^2 + 2\varepsilon\langle X, \mathsf{h}_{H,t}(\mathrm{d} f)\rangle + \varepsilon^2|\mathsf{h}_{H,t}(\mathrm{d} f)|^2 \leq c^2 + 2\varepsilon \mathsf{h}_t\langle X, \nabla f\rangle + \varepsilon^2 \mathsf{h}_t(|\mathrm{d} f|^2)$$

and the arbitrariness of $\varepsilon \in \mathbb{R}$ implies

$$\langle X, \mathsf{h}_{H,t}\mathrm{d} f\rangle = 2\mathsf{h}_t\langle X, \nabla f\rangle,$$

which by (1.4.8.12) is (3.3.0.4). Then (3.3.0.5) comes by differentiating (3.3.0.4) at $t = 0$. □

Proposition 3.3.4 (Preservation of the Dirichlet energy)**.** *Let* $(\mathrm{M},\mathsf{d},\mathfrak{m})$ *be a* $\mathsf{RCD}(0,\infty)$ *space and* X *a harmonic vector field. Then for every* $t \in \mathbb{R}$ *we have*

$$\mathsf{E}(f \circ \mathrm{Fl}_t^X) = \mathsf{E}(f) \qquad \forall f \in W^{1,2}(\mathrm{M}). \tag{3.3.0.7}$$

proof Fix $f \in W^{1,2}(\mathrm{M})$, put $f_t := f \circ \mathrm{Fl}_t^X$ and notice that since $\mathsf{E}(\mathsf{h}_\varepsilon g) \to \mathsf{E}(g)$ as $\varepsilon \downarrow 0$ for any $g \in L^2(\mathrm{M})$, it is sufficient to prove that for any $\varepsilon > 0$ we have

$$\mathsf{E}(\mathsf{h}_\varepsilon f_t) = \mathsf{E}(\mathsf{h}_\varepsilon f) \qquad \forall t \in \mathbb{R}.$$

Thus fix $\varepsilon > 0$ and notice that Proposition 2.5.7 grants that $t \mapsto f_t \in L^2(\mathrm{M})$ is Lipschitz. This in conjunction with the fact that $\mathsf{h}_\varepsilon : L^2(\mathrm{M}) \to W^{1,2}(\mathrm{M})$ is continuous ensures that $t \mapsto \mathsf{h}_\varepsilon f_t \in W^{1,2}(\mathrm{M})$ is Lipschitz.

We now compute the derivative of the Lipschitz map $t \mapsto \mathsf{E}(\mathsf{h}_\varepsilon f_t)$ and start noticing that

$$\int |\nabla \mathsf{h}_\varepsilon f_{t+h}|^2 - |\nabla \mathsf{h}_\varepsilon f_t|^2 \mathrm{d}\mathfrak{m} = \int |\nabla \mathsf{h}_\varepsilon (f_{t+h} - f_t)|^2 + 2\big\langle \nabla \mathsf{h}_\varepsilon f_t, \nabla \mathsf{h}_\varepsilon (f_{t+h} - f_t)\big\rangle \mathrm{d}\mathfrak{m},$$

so that the Lipschitz regularity of $t \mapsto \mathsf{h}_\varepsilon f_t \in W^{1,2}(\mathrm{M})$ grants that for any $t \in \mathbb{R}$ it holds

$$\lim_{h\to 0} \int \frac{|\nabla \mathsf{h}_\varepsilon f_{t+h}|^2 - |\nabla \mathsf{h}_\varepsilon f_t|^2}{2h} \mathrm{d}\mathfrak{m} = \lim_{h\to 0} \int \Big\langle \nabla \mathsf{h}_\varepsilon f_t, \nabla \frac{\mathsf{h}_\varepsilon f_{t+h} - \mathsf{h}_\varepsilon f_t}{h} \Big\rangle \mathrm{d}\mathfrak{m}.$$

Hence

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \mathsf{E}(\mathsf{h}_\varepsilon f_t) &= -\lim_{h\to 0} \int \Delta \mathsf{h}_\varepsilon f_t \ \frac{\mathsf{h}_\varepsilon f_{t+h} - \mathsf{h}_\varepsilon f_t}{h} \mathrm{d}\mathfrak{m} \\
&= -\lim_{h\to 0} \int \Delta \mathsf{h}_{2\varepsilon} f_t \ \frac{f_t \circ \mathrm{Fl}_h^{(X)} - f_t}{h} \mathrm{d}\mathfrak{m} \\
\text{by (3.3.0.3)} \qquad &= -\lim_{h\to 0} \int \frac{\big(\Delta \mathsf{h}_{2\varepsilon} f_t\big) \circ \mathrm{Fl}_{-h}^{(X)} - \Delta \mathsf{h}_{2\varepsilon}(f_t)}{h} f_t \mathrm{d}\mathfrak{m} \\
\text{by the last claim in Proposition 2.5.7} \qquad &= -\int \langle \nabla \Delta \mathsf{h}_{2\varepsilon} f_t, X \rangle \, f_t \mathrm{d}\mathfrak{m}.
\end{aligned}$$

To conclude it is therefore sufficient to prove that for any $g \in L^2(\mathrm{M})$ it holds

$$\int \langle \nabla \Delta \mathsf{h}_{2\varepsilon} g, X \rangle \, g \, \mathrm{d}\mathfrak{m} = 0 \tag{3.3.0.8}$$

Hence fix $g \in L^2(\mathrm{M})$ and notice that

$$\int \langle \nabla \Delta \mathsf{h}_{2\varepsilon} g, X \rangle \, g \, \mathrm{d}\mathfrak{m} \overset{(3.3.0.4)}{=} \int \mathsf{h}_\varepsilon \langle \nabla \Delta \mathsf{h}_\varepsilon g, X \rangle g \, \mathrm{d}\mathfrak{m} = \int \langle \nabla \Delta \mathsf{h}_\varepsilon g, X \rangle \mathsf{h}_\varepsilon g \, \mathrm{d}\mathfrak{m} \tag{3.3.0.9}$$

and, recalling that $\mathrm{div} X = 0$, that

$$\int \langle \nabla \Delta \mathsf{h}_\varepsilon g, X \rangle \, \mathsf{h}_\varepsilon g \, \mathrm{d}\mathfrak{m} = -\int \Delta \mathsf{h}_\varepsilon g \, \langle X, \nabla \mathsf{h}_\varepsilon g \rangle \, \mathrm{d}\mathfrak{m} \overset{(3.3.0.5)}{=} -\int \mathsf{h}_\varepsilon g \, \langle X, \nabla \Delta \mathsf{h}_\varepsilon g \rangle \, \mathrm{d}\mathfrak{m}.$$

This proves (3.3.0.8) and the theorem. □

We therefore can conclude that:

Theorem 3.3.5. *Let* $(\mathrm{M}, \mathsf{d}, \mathfrak{m})$ *be a* $\mathsf{RCD}(0,\infty)$ *space and* X *a harmonic vector field. Then for every* $t \in \mathbb{R}$ *the map* $\mathrm{Fl}_t^{(X)}$ *has a continuous representative and this representative is a measure preserving isometry.*

proof Use the preservation of measure proved in Proposition 3.3.2 and the one of Dirichlet energy proved in Proposition 3.3.4 in conjunction with Theorem 1.4.3. □

From now on we shall identify $\mathrm{Fl}_t^{(X)}$ with its continuous representative. It is readily verified from the construction that the group property 3.3.0.2 holds everywhere.

One of the consequences of the fact that $\mathrm{Fl}_t^{(X)}$ is an automorphism of $(\mathrm{M}, \mathsf{d}, \mathfrak{m})$ is that

$$f \in \mathrm{Test}(\mathrm{M}) \qquad \Rightarrow \qquad f \circ \mathrm{Fl}_t^{(X)} \in \mathrm{Test}(\mathrm{M}). \tag{3.3.0.10}$$

This can be seen by noticing that since $\mathrm{Fl}_t^{(X)}$ is a measure preserving isometry, directly from the definition of Sobolev space we have

$$f \in W^{1,2}(\mathrm{M}) \quad \Leftrightarrow \quad f \circ \mathrm{Fl}_t^{(X)} \in W^{1,2}(\mathrm{M}) \quad \text{and in this case } |\mathrm{d}f| \circ \mathrm{Fl}_t^{(X)} = |\mathrm{d}(f \circ \mathrm{Fl}_t^{(X)})|.$$

From this fact and the definition of Laplacian we then deduce that

$$f \in D(\Delta) \quad \Leftrightarrow \quad f \circ \mathrm{Fl}_t^{(X)} \in D(\Delta) \quad \text{and in this case } (\Delta f) \circ \mathrm{Fl}_t^{(X)} = \Delta(f \circ \mathrm{Fl}^{(X)}).$$

A suitable iteration of these arguments then yields (3.3.0.10).

Recall also (see [36]) that being Fl_t^X invertible and of bounded deformation, its differential dFl_t^X is a map from $L^2(T\mathrm{M})$ into itself (well) defined by:

$$\mathrm{d}f(\mathrm{d}\,\mathrm{Fl}_s^X(Y)) = \mathrm{d}(f \circ \mathrm{Fl}_s^X)(Y) \circ \mathrm{Fl}_{-s}^X \qquad \forall f \in W^{1,2}(\mathrm{M}). \tag{3.3.0.11}$$

We now want to prove that if X, Y are both harmonic, their flows commute. The proof is based on the following lemma:

Lemma 3.3.6. *Let* $(\mathrm{M}, \mathsf{d}, \mathfrak{m})$ *be a* $\mathsf{RCD}(0,\infty)$ *space and* X, Y *harmonic vector fields. Then*

$$\mathrm{d}\,\mathrm{Fl}_s^{(X)}(Y) = Y \qquad \forall s \in \mathbb{R}.$$

proof Since differential of test functions generate the whole cotangent module, the claim will follow if we show that for any $f \in \mathrm{Test}(\mathrm{M})$ the map $\mathbb{R} \ni s \mapsto \mathrm{d}f(\mathrm{dFl}_s^{(X)}(Y))$ is constant.

Taking account of the equality in (3.3.0.11) and recalling (3.3.0.10), in order to conclude it is sufficient to prove that for any $f \in \mathrm{Test}(\mathrm{M})$

$$\frac{\mathrm{d}(f \circ \mathrm{Fl}_h^{(X)})(Y) \circ \mathrm{Fl}_{-h}^{(X)} - \mathrm{d}f(Y)}{h} \quad \text{goes to 0 in the strong } L^2(\mathrm{M})\text{-topology as } h \to 0.$$

To this aim, start observing that

$$\frac{\mathrm{d}(f \circ \mathrm{Fl}_h^{(X)})(Y) \circ \mathrm{Fl}_{-h}^{(X)} - \mathrm{d}f(Y)}{h} = \mathrm{d}\bigg(\frac{f \circ \mathrm{Fl}_h^{(X)} - f}{h}\bigg)(Y) \circ \mathrm{Fl}_{-h}^{(X)} + \frac{\mathrm{d}f(Y) \circ \mathrm{Fl}_{-h}^{(X)} - \mathrm{d}f(Y)}{h}.$$

Since $f \in \mathrm{Test}(\mathrm{M})$ and $X \in H_C^{1,2}(T\mathrm{M})$ we have $\mathrm{d}f(Y) \in W^{1,2}(\mathrm{M})$ (see [36] for details about this implication) and thus from the last claim in Proposition (2.5.7) we have

$$\lim_{s \to 0} \frac{\mathrm{d}f(Y) \circ \mathrm{Fl}_{-s}^{(X)} - \mathrm{d}f(Y)}{s} = -\mathrm{d}(\mathrm{d}f(Y))(X) \qquad \text{in } L^2(\mathrm{M}),$$

hence to conclude it is sufficient to show that

$$\lim_{s \to 0} \mathrm{d}\bigg(\frac{f \circ \mathrm{Fl}_s^{(X)} - f}{s}\bigg)(Y) \circ \mathrm{Fl}_{-s}^{(X)} = \mathrm{d}(\mathrm{d}f(X))(Y). \tag{3.3.0.12}$$

Let us start proving that

$$\frac{f \circ \mathrm{Fl}_s^{(X)} - f}{s} \quad \to \quad \mathrm{d}f(X) \quad \text{as } s \to 0 \qquad \text{in } W^{1,2}(\mathrm{M}). \tag{3.3.0.13}$$

Notice that (2.5.0.12) grants convergence in $L^2(\mathrm{M})$; moreover the bound

$$\begin{aligned}\Big|\mathrm{d}\Big(\frac{f\circ \mathrm{Fl}^{(X)}_s - f}{s}\Big)\Big|^2 &= \Big|\frac{1}{s}\int_0^s \mathrm{d}\big(\mathrm{d}f(X)\circ \mathrm{Fl}^{(X)}_r\big)\,\mathrm{d}r\Big|^2\\ &\le \frac{1}{s}\int_0^s |\mathrm{d}\big(\mathrm{d}f(X)\circ \mathrm{Fl}^{(X)}_r\big)|^2\,\mathrm{d}r = \frac{1}{s}\int_0^s |\mathrm{d}(\mathrm{d}f(X))|^2\circ \mathrm{Fl}^{(X)}_r\,\mathrm{d}r\end{aligned}$$

and the fact that $(\mathrm{Fl}^X_r)_*\mathfrak{m}=\mathfrak{m}$ grant that $\overline{\lim}_{s\to 0}\|\frac{f\circ\mathrm{Fl}^{(X)}_s - f}{s}\|_{W^{1,2}(\mathrm{M})}\le \|\mathrm{d}f(X)\|_{W^{1,2}(\mathrm{M})}$ which is sufficient to get (4.4.0.3).

From (4.4.0.3) we deduce that

$$\mathrm{d}\Big(\frac{f\circ \mathrm{Fl}^{(X)}_s - f}{s}\Big)(Y)\quad\to\quad \mathrm{d}(\mathrm{d}f(X))(Y)\quad\text{as } s\to 0\qquad \text{in } L^2(\mathrm{M})$$

hence (3.3.0.12) follows from (2.5.0.12). □

Theorem 3.3.7. *Let* $(\mathrm{M},\mathsf{d},\mathfrak{m})$ *be a* $\mathsf{RCD}(0,\infty)$ *space and* $X,Y\in L^2(T\mathrm{M})$ *be two harmonic vector fields. Then for any* $t,s\in\mathbb{R}$ *it holds*

$$\mathrm{Fl}^X_t\circ \mathrm{Fl}^Y_s = \mathrm{Fl}^{tX+sY}_1.$$

proof For any $r\in\mathbb{R}$ consider the map $G_r := \mathrm{Fl}^X_{rt}\circ\mathrm{Fl}^Y_{rs}$. Now take $f\in W^{1,2}(\mathrm{M})$ and observe that $f\circ\mathrm{Fl}^X_{rt}\in W^{1,2}(\mathrm{M})$, as a consequence of Theorem 3.3.5, and that from Proposition 2.5.7 it easily follows that $r\mapsto f\circ G_r\in L^2(\mathrm{M})$ is Lipschitz. By direct computation we have:

$$\frac{\mathrm{d}}{\mathrm{d}r}(f\circ G_r) = \frac{\mathrm{d}}{\mathrm{d}r}(f\circ \mathrm{Fl}^X_{rt}\circ\mathrm{Fl}^Y_{rs}) = s\,\mathrm{d}(f\circ\mathrm{Fl}^X_{rt})(Y)\circ\mathrm{Fl}^Y_{rs} + t\,\mathrm{d}f(X)\circ\mathrm{Fl}^X_{rt}\circ\mathrm{Fl}^Y_{rs}. \tag{3.3.0.14}$$

Using first identity (3.3.0.11) and then Lemma 3.3.6 we have

$$\mathrm{d}(f\circ\mathrm{Fl}^X_{rt})(Y)\circ\mathrm{Fl}^Y_{rs} = \mathrm{d}f(\mathrm{d}\mathrm{Fl}^X_{rt}(Y))\circ\mathrm{Fl}^X_{rt}\circ\mathrm{Fl}^Y_{rs} = \mathrm{d}f(Y)\circ\mathrm{Fl}^X_{rt}\circ\mathrm{Fl}^Y_{rs}$$

and thus from (3.3.0.14) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}(f\circ G_r) = s\,\mathrm{d}f(Y)\circ\mathrm{Fl}^X_{rt}\circ\mathrm{Fl}^Y_{rs} + t\,\mathrm{d}f(X)\circ\mathrm{Fl}^X_{rt}\circ\mathrm{Fl}^Y_{rs} = \mathrm{d}f(tX+sY)\circ G_r$$

and since it is obvious by construction that (G_r) has the properties $(i),(ii)$ in Definition 2.3.3, by Proposition 2.5.7 we deduce that (G_r) is a Regular Lagrangian Flow of $tX+sY$ and thus by the uniqueness part of Theorem 2.5.5 we deduce that for any $r\ge 0$ we have

$$\mathrm{Fl}^X_{rt}\circ\mathrm{Fl}^Y_{rs} = \mathrm{Fl}^{tX+sY}_r$$

$\mathfrak{m}$-a.e.. In particular this holds for $r=1$ and since both sides are continuous functions, equality holds everywhere. □

3.4 Proof of the Bochner Theorem

3.4.1 Setting and preliminary results

First of all we introduce the setting where we are going to work as well as the assumptions and notations we are going to use.

Setting: Let $(\mathrm{M}, \mathsf{d}, \mathfrak{m})$ be a $\mathsf{RCD}^*(0, N)$ metric measure space with $\mathrm{supp}(\mathfrak{m}) = \mathrm{M}$, $N \in \mathbb{N}$, $N > 0$, and such that $\dim(\mathcal{H}^1_{\mathrm{dR}})(\mathrm{M}) = N$. By Theorem 1.4.35 this hypothesis implies that there exists N harmonic vector fields $X_1, \ldots, X_N$ which are orthogonal in $L^2(T\mathrm{M})$. In particular, this means that the vector fields X_i's span the whole tangent space, since $\mathsf{dim}_{\min}(\mathrm{M}) \leq \mathsf{dim}_{\max}(\mathrm{M}) \leq N$, as proved in [45], and so this set of independent vector fields actually forms a basis (of dimension N) on $L^2(T\mathrm{M})$. Moreover, as we have seen in Section 3.3, the fact that the Ricci curvature is non-negative implies that these vector fields belong to $H^{1,2}_C(T\mathrm{M})$ and are parallel, i.e. $\nabla X_i \equiv 0$ for every i. It follows that $\langle X_i, X_j\rangle \in W^{1,2}(\mathrm{M})$ with

$$\mathrm{d}\langle X_i, X_j\rangle = \nabla X_i(\cdot, X_j) + \nabla X_j(\cdot, X_i) = 0 \qquad \mathfrak{m} - a.e.,$$

which in turn grants that $\langle X_i, X_j\rangle$ is $\mathfrak{m}$-a.e. equal to a constant function. Since $\int \langle X_i, X_j\rangle \,\mathrm{d}\mathfrak{m} = 0$ for $i \neq j$ we conclude that the X_i's are pointwise orthogonal. The same argument also shows that up to normalization we can, and will, assume that $|X_i| \equiv 1$ $\mathfrak{m}$-a.e. for every i. In particular, since these vector fields are in $L^2(T\mathrm{M})$, we have

$$\mathfrak{m}(\mathrm{M}) < \infty. \tag{3.4.1.1}$$

In particular in the following we shall work in the product space $\mathrm{M} \times \mathbb{R}^N$ which will be equipped with the measure $\mathfrak{m} \times \mathcal{L}^N$ and the distance

$$(\mathsf{d} \otimes \mathsf{d}_{\mathrm{Eucl}})^2\big((x, a), (y, b)\big) := \mathsf{d}^2(x, y) + |a - b|^2.$$

Recall that in [10] (see also [9]) it has been proved that the product of two $\mathsf{RCD}(0, \infty)$ spaces is also $\mathsf{RCD}(0, \infty)$, so that $\mathrm{M} \times \mathbb{R}^N$ is $\mathsf{RCD}(0, \infty)$.

Definition of the vector fields Y_i and their properties: At this point we define the vector fields $Y_i \in L^0(\mathrm{M} \times \mathbb{R}^N)$, $i = 1, \ldots, N$ as

$$Y_i := \Phi_2(\widehat{\nabla\pi_i}) \qquad \forall i = 1, \ldots, N,$$

where $\pi_i \colon \mathbb{R}^N \to \mathbb{R}$ is the projection on the i-th coordinate, $\widehat{\nabla\pi_i} \in L^0(\mathrm{M}, L^0(T\mathbb{R}^N))$ is the function identically equal to $\nabla\pi_i \in L^0(T\mathbb{R}^N)$ and where $\Phi_2 \colon L^0(\mathrm{M}, L^0(T\mathbb{R}^N)) \to L^0(T(\mathrm{M} \times \mathbb{R}^N))$ is defined in Proposition 3.2.1. Intuitively, this means that each one of the vector fields Y_i plays in $\mathrm{M} \times \mathbb{R}^N$ the same role that the vector field $\nabla\pi_i$ does in $\mathbb{R}^N$.

From the fact that the $\nabla\pi_i$ are a pointwise ortonormal base for $L^0(T\mathbb{R}^N)$ and the fact that Φ_2 preserves the pointwise norm we deduce that

$$\langle Y_i, Y_j\rangle = \delta_{ij} \quad \mathfrak{m} \times \mathcal{L}^N - a.e. \quad \forall i, j$$

and from the very definition of Φ_2 we have that

$$Y_i = \nabla(\pi_i \circ \pi^{\mathbb{R}^N}). \tag{3.4.1.2}$$

Since $\pi_i : \mathbb{R}^N \to \mathbb{R}$ is harmonic, we have $\nabla\pi_i \in D(\mathrm{div}_{\mathsf{loc}}, \mathbb{R}^N)$ with $\mathrm{div}(\nabla\pi_i) = 0$ and thus from Propositions 3.2.9 and 3.2.5 we deduce that $Y_i \in D(\mathrm{div}_{\mathsf{loc}}, \mathrm{M} \times \mathbb{R}^N)$ with

$$\mathrm{div} Y_i \equiv 0. \tag{3.4.1.3}$$

Taking into account (1.4.7.13) we also obtain that $Y_i^\flat \in D(\Delta_{H,\mathsf{loc}})$ with $\Delta_H Y_i^\flat = 0$ and since $|\nabla\pi_i| \equiv 1$ and Φ_2 preserves the pointwise norm we also deduce that $|Y_i| \equiv 1$: these facts together with (1.4.8.1) grant that $Y_i \in W^{1,2}_{C,\mathsf{loc}}(T(\mathrm{M} \times \mathbb{R}^N))$ with

$$\nabla Y_i \equiv 0. \tag{3.4.1.4}$$

Definition of the map T and its properties: Finally, using the Regular Lagrangian flows of the vector fields $X_1, \ldots, X_N$, we can define the map

$$\begin{aligned} \mathsf{T} \colon \mathrm{M} \times \mathbb{R}^N &\to \mathrm{M} \\ (x, \underline{a} = (a_1, \ldots a_N)) &\mapsto \mathrm{Fl}^{(X_1)}_{a_1} \circ \cdots \circ \mathrm{Fl}^{(X_N)}_{a_N}(x). \end{aligned} \tag{3.4.1.5}$$

From Theorem 3.3.7 we deduce that

$$\mathsf{T}(\mathsf{T}(x, \underline{a}), \underline{b}) = \mathsf{T}(x, \underline{a} + \underline{b}) \qquad \forall x \in \mathrm{M},\ \underline{a}, \underline{b} \in \mathbb{R}^N, \tag{3.4.1.6}$$

so that it is reasonable to think at T as an action of $\mathbb{R}^N$ on M. Theorem 3.3.5 grants that this action is made of isometries, i.e.

$$\mathsf{T}(\cdot, \underline{a}) \colon \mathrm{M} \to \mathrm{M} \quad \text{ is an isometry for any } \underline{a} \in \mathbb{R}^N. \tag{3.4.1.7}$$

From Theorem 3.3.7 we also have

$$\mathsf{T}(x, \underline{a}) = \mathrm{Fl}^{(X_{\underline{a}})}_1(x) \qquad \text{for} \qquad X_{\underline{a}} := \sum_{i=1}^N a_i X_i$$

and since the pointwise orthonormality of the X_i's gives $|X_{\underline{a}}|^2 = |\sum_{i=1}^N a_i X_i|^2 = \sum_i |a_i|^2 = |\underline{a}|^2$ $\mathfrak{m}$-a.e., from (2.5.0.9) we deduce that for $\mathfrak{m}$-a.e. x it holds

$$\mathsf{d}(x, \mathsf{T}(x, \underline{a})) \leq \int_0^1 |X_{\underline{a}}| \circ \mathrm{Fl}^{(X_{\underline{a}})}_t \,\mathrm{d}t \leq \||X_{\underline{a}}|\|_{L^\infty} = |\underline{a}|.$$

Now the continuity of $\mathsf{T}(\cdot, \underline{a})$ ensures that the above holds for every $x \in \mathrm{M}$ and thus taking (3.4.1.6) into account we conclude that

$$\mathsf{T}(x, \cdot) \colon \mathbb{R}^N \to \mathrm{M} \quad \text{ is 1-Lipschitz for any } x \in \mathrm{M}. \tag{3.4.1.8}$$

Finally we remark that Proposition 3.3.2 together with Fubini theorem guarantees that

$$\mathsf{T}_*(\mathfrak{m} \times \mathscr{L}^N|_A) = \mathscr{L}^N(A)\mathfrak{m}, \tag{3.4.1.9}$$

for every $A \subset \mathbb{R}^N$ Borel. This identity, (3.4.1.7) and (3.4.1.8) grant in particular that $\mathsf{T} : \mathrm{M} \times \mathbb{R}^N \to \mathrm{M}$ is of local bounded deformation.

3.4.2 An explicit formula for Regular Lagrangian Flows on M

Proposition 3.4.1 (Conjugation property). *With the same notations and assumptions as in Section 3.4.1 we have that for any $i = 1, \dots, N$ and $f \in W^{1,2}_{\mathsf{loc}}(\mathrm{M})$ it holds*

$$\mathrm{d}(f \circ \mathsf{T})(Y_i) = \mathrm{d}f(X_i) \circ \mathsf{T} \qquad \mathfrak{m} \times \mathcal{L}^N - a.e.. \tag{3.4.2.1}$$

proof For any $t \in \mathbb{R}$ let us set $\bar{\mathrm{Fl}}^i_t(x, \underline{a}) := (x, \underline{a} + te_i)$ and notice that by the very definition of T and identity(3.4.1.6) we have

$$\mathrm{Fl}^{X_i}_t \circ \mathsf{T} = \mathsf{T} \circ \bar{\mathrm{Fl}}^i_t. \tag{3.4.2.2}$$

Fix $f \in W^{1,2}_{\mathsf{loc}}(\mathrm{M})$ and recall Proposition 2.5.7 to get

$$\mathrm{d}f(X_i) \circ \mathsf{T} = \Big(\lim_{t \downarrow 0} \frac{f \circ \mathrm{Fl}^{(X_i)}_t - f}{t} \Big) \circ \mathsf{T} \overset{(3.4.2.2)}{=} \lim_{t \downarrow 0} \frac{f \circ \mathsf{T} \circ \bar{\mathrm{Fl}}^i_t - f \circ \mathsf{T}}{t},$$

the first limit being in $L^2_{\mathsf{loc}}(\mathrm{M})$ and the second in $L^2_{\mathsf{loc}}(\mathrm{M} \times \mathbb{R}^N)$. Hence to conclude it is sufficient to show that for any $\tilde{f} \in W^{1,2}_{\mathsf{loc}}(\mathrm{M} \times \mathbb{R}^N)$ and $\rho \in L^\infty(\mathrm{M} \times \mathbb{R}^N)$ with bounded support we have

$$\lim_{t \downarrow 0} \int \frac{\tilde{f} \circ \bar{\mathrm{Fl}}^i_t - \tilde{f}}{t} \rho \, \mathrm{d}(\mathfrak{m} \times \mathcal{L}^N) = \int \mathrm{d}\tilde{f}(Y_i) \rho \, \mathrm{d}(\mathfrak{m} \times \mathcal{L}^N). \tag{3.4.2.3}$$

By the linearity in ρ of this expression we can further assume that ρ is a probability density. Then put $\mu := \rho\mathfrak{m}$ and $\boldsymbol{\pi} := (\bar{\mathrm{Fl}}^i_\cdot)_* \mu$, where here $\bar{\mathrm{Fl}}^i_\cdot : \mathrm{M} \times \mathbb{R}^N \to C([0,1], \mathrm{M} \times \mathbb{R}^N)$ is the map sending $(x, \underline{a})$ to the curve $[0,1] \ni t \mapsto \mathrm{Fl}^i_t(x, \underline{a})$. Notice that $\boldsymbol{\pi}$ is a test plan on $\mathrm{M} \times \mathbb{R}^N$ which is concentrated on curves with speed constantly equal to 1, thus for any $\tilde{f} \in W^{1,2}_{\mathsf{loc}}(\mathrm{M} \times \mathbb{R}^N)$ we have

$$\begin{aligned}
\frac{\int \tilde{f} \, \mathrm{d}(\bar{\mathrm{Fl}}^i_t)_* \mu - \int \tilde{f} \, \mathrm{d}\mu}{t} &= \frac{1}{t} \int \tilde{f}(\gamma_t) - \tilde{f}(\gamma_0) \, \mathrm{d}\boldsymbol{\pi}(\gamma) \\
&\leq \frac{1}{t} \iint_0^t |\mathrm{d}\tilde{f}|(\gamma_s) |\dot{\gamma}_s| \, \mathrm{d}s \, \mathrm{d}\boldsymbol{\pi}(\gamma) \\
&\leq \frac{1}{2t} \iint_0^t |\mathrm{d}\tilde{f}|^2(\gamma_s) \, \mathrm{d}s \, \mathrm{d}\boldsymbol{\pi}(\gamma) + \frac{1}{2t} \iint_0^t |\dot{\gamma}^2_s| \, \mathrm{d}s \, \mathrm{d}\boldsymbol{\pi}(\gamma) \\
&= \frac{1}{2t} \int_0^t \int |\mathrm{d}\tilde{f}|^2 \circ \bar{\mathrm{Fl}}^i_s \, \mathrm{d}\mu \, \mathrm{d}s + \frac{1}{2}.
\end{aligned}$$

Recalling (2.5.0.12) we thus have

$$\varlimsup_{t \downarrow 0} \frac{\int \tilde{f} \, \mathrm{d}(\bar{\mathrm{Fl}}^i_t)_* \mu - \int \tilde{f} \, \mathrm{d}\mu}{t} \leq \frac{1}{2} \int |\mathrm{d}\tilde{f}|^2 \, \mathrm{d}\mu + \frac{1}{2}. \tag{3.4.2.4}$$

Now put for brevity $f_i := \pi^i \circ \pi^{R^N}$, so that f_i is 1-Lipschitz, (3.4.1.2) reads as

$$\nabla f_i = Y_i \tag{3.4.2.5}$$

and by construction it holds $f_i \circ \bar{\mathrm{Fl}}^i_s = f_i + s$, so that

$$\varliminf_{t \downarrow 0} \frac{\int f_i \, \mathrm{d}(\bar{\mathrm{Fl}}^i_t)_* \mu - \int f_i \, \mathrm{d}\mu}{t} \geq 1 \geq \frac{1}{2} \int |\mathrm{d}f_i|^2 \, \mathrm{d}\mu + \frac{1}{2}. \tag{3.4.2.6}$$

(In the terminology of [38] we just proved that $\boldsymbol{\pi}$ represents the gradient of f_i and we are now going to use the link between 'horizontal and vertical' derivatives). Writing (3.4.2.4) for $f+\varepsilon\tilde f$ in place of $\tilde f$ and subtracting (3.4.2.6) we obtain

$$\overline{\lim_{t\downarrow 0}}\,\varepsilon\frac{\int \tilde f\,\mathrm{d}(\bar{\mathrm{Fl}}^i_t)_*\mu-\int\tilde f\,\mathrm{d}\mu}{t}\leq\frac12\int|\mathrm{d}(f+\varepsilon\tilde f)|^2-|\mathrm{d}f|^2\,\mathrm{d}\mu\overset{(3.4.2.5)}{=}\int\varepsilon\,\mathrm{d}\tilde f(Y_i)+\frac{\varepsilon^2}{2}|\mathrm{d}\tilde f|^2\,\mathrm{d}\mu.$$

Dividing by $\varepsilon>0$ (resp. $\varepsilon<0$) and letting $\varepsilon\downarrow 0$ (resp. $\varepsilon\uparrow 0$) we obtain (3.4.2.3) and the conclusion. □

We now introduce the map $\Psi : L^0(T\mathrm{M})\to L^0(T(\mathrm{M}\times\mathbb{R}^N))$ defined as

$$\Psi(v):=\sum_{i=1}^N\langle v,X_i\rangle\circ\mathsf{T}\;\,Y_i. \tag{3.4.2.7}$$

Lemma 3.4.2. *With the same notations and assumptions as in Section 3.4.1 and with Ψ defined as in* (3.4.2.7)*, the following holds. Let $v\in L^\infty\cap W^{1,2}_C(T\mathrm{M})$. Then:*

i) $\langle v,X_i\rangle\in W^{1,2}(\mathrm{M})$ *for every i and $v\in D(\mathrm{div})$ with*

$$\mathrm{div}(v)=\sum_{i=1}^N\mathrm{d}(\langle v,X_i\rangle)(X_i) \tag{3.4.2.8}$$

ii) $\Psi(v)\in L^\infty\cap W^{1,2}_{C,\mathsf{loc}}\cap D(\mathrm{div}_{\mathsf{loc}})(T(\mathrm{M}\times\mathbb{R}^n))$ *with*

$$\begin{aligned}\nabla(\Psi(v))&=\sum_{i=1}^N\nabla(\langle v,X_i\rangle\circ\mathsf{T})\otimes Y_i,\\ \mathrm{div}(\Psi(v))&=\mathrm{div}(v)\circ\mathsf{T}.\end{aligned}$$

proof
(i) From [36] we know that the assumptions on v grant that $\langle v,X\rangle\in W^{1,2}_{\mathsf{loc}}(\mathrm{M})$ for every $X\in L^\infty\cap H^{1,2}_C(T\mathrm{M})$ with

$$\mathrm{d}\langle v,X\rangle=\nabla v(\cdot,X)+\nabla X(\cdot,v).$$

Picking $X:=X_i$ and recalling that $D(\Delta_H)\subset(H^{1,2}_C(T\mathrm{M}))^\flat$ by the very definition of Δ_H, we conclude that $\langle v,X_i\rangle$ belongs to $W^{1,2}(\mathrm{M})$, as claimed. Now put $a_i:=\langle v,X_i\rangle$ for brevity, so that $v=\sum_i a_iX_i$, let $f\in W^{1,2}(\mathrm{M})$ and notice that

$$\int\mathrm{d}f(v)\,\mathrm{d}\mathfrak{m}=\sum_{i=1}^N\int\mathrm{d}f(a_iX_i)\,\mathrm{d}\mathfrak{m}=-\sum_{i=1}^N\int f\mathrm{div}(a_iX_i)=-\int\sum_{i=1}^N f\mathrm{d}a_i(X_i)\,\mathrm{d}\mathfrak{m},$$

having used the fact that $\mathrm{div}(X_i)=0$. This proves both $v\in D(\mathrm{div})$ and (3.4.2.8).
(ii) The assumption $v\in L^\infty(T\mathrm{M})$ trivially yields $a_i\in L^\infty(\mathrm{M})$ and since $Y_i\in L^\infty\cap W^{1,2}_C(\mathrm{M}\times\mathbb{R}^N)$ with $\nabla Y_i=0$ (recall (3.4.1.4)), we have $(a_i\circ\mathsf{T})\,Y_i\in W^{1,2}_{C,\mathsf{loc}}(T(\mathrm{M}\times\mathbb{R}^N))$ with

$$\nabla((a_i\circ\mathsf{T})\,Y_i)=\nabla(a_i\circ\mathsf{T})\otimes Y_i+a_i\circ\mathsf{T}\,\nabla Y_i=\nabla(a_i\circ\mathsf{T})\otimes Y_i.$$

The fact that $\Phi(v)\in L^\infty\cap W^{1,2}_{C,\mathsf{loc}}(T(\mathrm{M}\times\mathbb{R}^n)$ and the formula for $\nabla(\Psi(v))$ follow.

We turn to the divergence: for $g \in W^{1,2}(\mathrm{M}\times\mathbb{R}^N)$ with bounded support we have

$$\begin{aligned}
\int \mathrm{d}g(\Psi(v))\,\mathrm{d}(\mathfrak{m}\times\mathcal{L}^N) &= \sum_{i=1}^N \int \mathrm{d}g(a_i\circ\mathsf{T}\,Y_i)\,\mathrm{d}(\mathfrak{m}\times\mathcal{L}^N)\\
\text{because } \mathrm{div}(Y_i)=0 \qquad &= -\sum_{i=1}^N \int g\,\mathrm{d}(a_i\circ\mathsf{T})(Y_i)\,\mathrm{d}(\mathfrak{m}\times\mathcal{L}^N)\\
\text{by (3.4.2.1)} \qquad &= -\sum_{i=1}^N \int g\,\mathrm{d}a_i(X_i)\circ\mathsf{T}\,\mathrm{d}(\mathfrak{m}\times\mathcal{L}^N),
\end{aligned}$$

which by (3.4.2.8) is the conclusion. □

Let $(v_t)\in L^\infty([0,1],L^2(T\mathrm{M}))\cap L^2([0,1],W^{1,2}_C(T\mathrm{M}))$ be such that $(\mathrm{div}(v_t))\in L^\infty([0,1],L^\infty(\mathrm{M}))$, so that in particular the Regular Lagrangian Flow $(\mathrm{Fl}^{(v_t)}_s)$ is well defined. The integrability condition of (v_t) ensures that $(\langle v_t,X_i\rangle)\in L^\infty([0,1],L^2(\mathrm{M}))$ for every $i=1,\ldots,N$ and thus from (2.3.0.7) we see that $(\langle v_s,X_i\rangle\circ\mathrm{Fl}^{(v_t)}_s)\in L^\infty([0,1],L^2(\mathrm{M}))$ as well. Hence the functions

$$A_{i,t} := \int_0^t \langle v_s,X_i\rangle\circ\mathrm{Fl}^{(v_t)}_s\,\mathrm{d}s\in L^2(\mathrm{M}),\qquad t\in[0,1],\ i=1,\ldots,N, \tag{3.4.2.9}$$

are well defined. We then have the following result:

Lemma 3.4.3. *With the same assumptions and notation as in Section 3.4.1, let* $(v_t)\in L^\infty([0,1],L^2(T\mathrm{M}))\cap L^2([0,1],W^{1,2}_C(T\mathrm{M}))$ *be such that* $(\mathrm{div}(v_t))\in L^\infty([0,1],L^\infty(\mathrm{M}))$ *and define* Ψ *as in* (3.4.2.7).

Then the vector fields $\Psi(v_t)$ *satisfy the assumptions of Theorem 2.5.5 and for any* $s\in\mathbb{R}$ *the following identities hold* $\mathfrak{m}\times\mathcal{L}^N$*-a.e.:*

$$\mathsf{T}\circ\mathrm{Fl}^{(\Psi(v_t))}_s = \mathrm{Fl}^{(v_t)}_s\circ\mathsf{T}, \tag{3.4.2.10}$$
$$\pi^{\mathrm{M}}\circ\mathrm{Fl}^{(\Psi(v_t))}_s = \pi^{\mathrm{M}} \tag{3.4.2.11}$$
$$\pi_i\circ\pi^{\mathbb{R}^N}\circ\mathrm{Fl}^{(\Psi(v_t))}_s = \pi_i\circ\pi^{\mathbb{R}^N} + A_{i,s}\circ\mathsf{T}. \tag{3.4.2.12}$$

proof The fact that the $\Psi(v_t)$'s satisfy the assumptions of Theorem 2.5.5 is a direct consequence of the assumptions and Lemma 3.4.2. Also, from (3.4.2.1) we directly deduce

$$\mathrm{d}(f\circ\mathsf{T})(\Psi(v_t)) = \mathrm{d}f(v_t)\circ\mathsf{T}\qquad \mathfrak{m}\times\mathcal{L}^N - a.e. \tag{3.4.2.13}$$

for every $t\in[0,1]$ and $f\in W^{1,2}_{\mathsf{loc}}(\mathrm{M})$. Now pick $\bar\mu_0\in\mathscr{P}(\mathrm{M}\times\mathbb{R}^N)$ with bounded support and such that $\bar\mu_0\le C\mathfrak{m}\times\mathcal{L}^N$ for some $C>0$ and for $s\in\mathbb{R}$ define

$$\bar\mu_s := (\mathrm{Fl}^{(\Psi(v_t))}_s)_*\bar\mu_0\in\mathscr{P}(\mathrm{M}\times\mathbb{R}^N)\qquad\text{and}\qquad \mu_s := \mathsf{T}_*\bar\mu_s\in\mathscr{P}(\mathrm{M}).$$

We claim that (μ_s) solves the continuity equation with vector fields (v_s) in the sense of Definition 2.3.5 and start observing that, locally in s, the measures $\bar\mu_s,\mu_s$ have uniformly bounded density. Now pick $f\in W^{1,2}(\mathrm{M})$, so that $f\circ\mathsf{T}\in W^{1,2}_{\mathsf{loc}}(\mathrm{M}\times\mathbb{R}^N)$ and, from Proposition 2.5.7, $s\mapsto$

$\int f\,\mathrm{d}\mu_s = \int f \circ \mathsf{T} \circ (\mathrm{Fl}_s^{(\Psi(v_t))})\,\mathrm{d}\bar{\mu}_0$ is Lipschitz with

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\int f\,\mathrm{d}\mu_s &= \frac{\mathrm{d}}{\mathrm{d}s}\int f \circ \mathsf{T} \circ (\mathrm{Fl}_s^{(\Psi(v_t))})\,\mathrm{d}\bar{\mu}_0 \\
\text{by Proposition 2.5.7} \qquad &= \int \mathrm{d}(f\circ\mathsf{T})(\Psi(v_s)) \circ (\mathrm{Fl}_s^{(\Psi(v_t))})\,\mathrm{d}\bar{\mu}_0 \\
&= \int \mathrm{d}(f\circ\mathsf{T})(\Psi(v_s))\,\mathrm{d}\bar{\mu}_s \\
\text{by (3.4.2.13)} \qquad &= \int \mathrm{d}f(v_s)\circ\mathsf{T}\,\mathrm{d}\bar{\mu}_s \\
&= \int \mathrm{d}f(v_s)\,\mathrm{d}\mu_s.
\end{aligned}$$

This proves our claim. Hence by the representation formula in Theorem 2.5.6 we deduce that

$$\mathsf{T}_*(\mathrm{Fl}_s^{(\Psi(v_t))})_*\bar{\mu}_0 = (\mathrm{Fl}_s^{(v_t)})_*\mathsf{T}_*\bar{\mu}_0 \qquad \forall s\in\mathbb{R}$$

and from the arbitrariness of $\bar{\mu}_0$ and (3.3.0.1) identity (3.4.2.10) follows.

To prove (3.4.2.11) pick $\bar{\mu}_0, f$ and define $\bar{\mu}_s$ as above. Then we also put

$$\nu_s := \pi_*^{\mathrm{M}}\bar{\mu}_s \in \mathscr{P}(\mathrm{M}) \qquad \forall s\in\mathbb{R}$$

and notice that again the ν_s's have, locally in s, uniformly bounded densities and that it holds

$$\frac{\mathrm{d}}{\mathrm{d}s}\int f\,\mathrm{d}\nu_s = \frac{\mathrm{d}}{\mathrm{d}s}\int f\circ\pi^{\mathrm{M}}\,\mathrm{d}\bar{\mu}_s = \int \mathrm{d}(f\circ\pi^{\mathrm{M}})(\Psi(v_s))\,\mathrm{d}\bar{\mu}_s = \int \Phi_1(\widehat{\mathrm{d}f})(\Psi(v_s))\,\mathrm{d}\bar{\mu}_s = 0,$$

where as usual $\widehat{\mathrm{d}f} \in L^0(\mathbb{R}^N, L^0(T^*\mathrm{M}))$ is the function identically equal to $\mathrm{d}f$ and the last identity follows from (3.2.2.1) and the very definitions of $\Psi(v_t)$ and Y_i. This shows that (ν_s) solves the continuity equation (2.3.0.10) with 0 vector fields, hence by the uniqueness of the solutions we deduce that (ν_s) is constant, i.e.

$$\pi_*^{\mathrm{M}}(\mathrm{Fl}_s^{(\Psi(v_t))})_*\bar{\mu}_0 = \pi_*^{\mathrm{M}}\bar{\mu}_0,$$

so that again the arbitrariness of $\bar{\mu}_0$ and (3.3.0.1) give (3.4.2.11).

For (3.4.2.12), we notice that the two sides agree for $s=0$ and are absolutely continuous as functions of s with values in $L^2_{\mathsf{loc}}(\mathrm{M}\times\mathbb{R}^N)$ (recall Proposition 2.5.7). The conclusion then follows recalling that it holds $\nabla(\pi_i\circ\pi^{\mathbb{R}^N}) = Y_i$, so that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\pi_i\circ\pi^{\mathbb{R}^N}\circ\mathrm{Fl}_s^{(\Psi(v_t))} &= \mathrm{d}(\pi_i\circ\pi^{\mathbb{R}^N})(\Psi(v_s))\circ\mathrm{Fl}_s^{(\Psi(v_t))} = \langle Y_i, \Psi(v_s)\rangle\circ\mathrm{Fl}_s^{(\Psi(v_t))} \\
&= \langle v_s, X_i\rangle\circ\mathsf{T}\circ\mathrm{Fl}_s^{(\Psi(v_t))} \overset{(3.4.2.10)}{=} \langle v_s, X_i\rangle\circ\mathrm{Fl}_s^{(v_t)}\circ\mathsf{T} = \frac{\mathrm{d}}{\mathrm{d}s}A_{i,s}\circ\mathsf{T}.
\end{aligned}$$

This is sufficient to conclude. □

We can now state the key result of this section:

Proposition 3.4.4 (Representation formula for $\mathrm{Fl}^{(v_t)}$)**.** *With the same assumptions and notation as in Section 3.4.1, let $(v_t) \in L^\infty([0,1], L^2(T\mathrm{M})) \cap L^2([0,1], W^{1,2}_C(T\mathrm{M}))$ be such that $(\mathrm{div}(v_t)) \in L^\infty([0,1], L^\infty(\mathrm{M}))$ and define the functions $A_{i,t}\in L^2(\mathrm{M})$ as in* (3.4.2.9)*.*

Then for any $s\in[0,1]$ and $\mathfrak{m}$-a.e. $x\in\mathrm{M}$ it holds

$$\mathrm{Fl}_s^{(v_t)}(x) = \mathsf{T}(x, \underline{A}_s(x)), \tag{3.4.2.14}$$

where $\underline{A}_s := (A_{1,s},\dots,A_{N,s})$.

proof The identities (3.4.2.11), (3.4.2.12) give

$$\mathrm{Fl}_s^{(\Psi(v_t))}(x,\underline{a}) = \big(x,\underline{a}+\underline{A}_s(\mathsf{T}(x,\underline{a}))\big) \qquad \mathfrak{m}\times\mathcal{L}^N - a.e.\ (x,\underline{a}).$$

Applying T on both sides and taking into account (3.4.2.10) and (3.4.1.6) we obtain

$$\mathrm{Fl}_s^{(v_t)}(\mathsf{T}(x,\underline{a})) = \mathsf{T}\big(\mathsf{T}(x,\underline{a}),\underline{A}_s(\mathsf{T}(x,\underline{a}))\big) \qquad \mathfrak{m}\times\mathcal{L}^N - a.e.\ (x,\underline{a}). \tag{3.4.2.15}$$

Thus for any $A\subset\mathbb{R}^N$ Borel we have that (3.4.2.14) holds for $\mathsf{T}_*(\mathfrak{m}\times\mathcal{L}^N|_A)$-a.e. $x\in\mathrm{M}$ and the conclusion follows from (3.4.1.9). □

3.4.3 Other properties of T and conclusion

First of all we need the following result, proved in [43], about W_2-geodesics and continuity equation. Recall that a measure $\boldsymbol{\pi}\in\mathscr{P}(C([0,1],\mathrm{M}))$ is called *lifting* of the geodesic (μ_t) provided

$$(\mathrm{e}_t)_*\boldsymbol{\pi} = \mu_t \qquad \forall t\in[0,1],$$
$$\iint_0^1 |\dot\gamma_t|^2\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma) < +\infty$$

and that, on an $\mathsf{RCD}^*(K,N)$ space, as soon as either μ_0 or μ_1 is absolutely continuous w.r.t. $\mathfrak{m}$, there is a unique geodesic connecting them and a unique lifting of it (see [44]).

Proposition 3.4.5. *Let* $(\mathrm{M},\mathsf{d},\mathfrak{m})$ *be a* $\mathsf{RCD}^*(K,N)$ *space and* $(\mu_t)\subset\mathscr{P}(\mathrm{M})$ *be a* W_2*-geodesic such that* μ_0,μ_1 *have both bounded support and density. Then there are vector fields* $(v_t)\subset L^2(\mathrm{M})$ *such that:*

i) the continuity equation (2.3.0.10) *is satisfied for* (μ_t,v_t) *in the sense of Definition* (2.3.5)

ii) letting $\boldsymbol{\pi}\in\mathscr{P}(C([0,1],\mathrm{M}))$ *be the lifting of* (μ_t) *it holds*

$$|v_t|(\gamma_t) = |\dot\gamma_t| \qquad \boldsymbol{\pi}\times\mathcal{L}^1 - a.e.\ (\gamma,t), \tag{3.4.3.1}$$

iii) $v_t\in H^{1,2}\cap D(\mathrm{div})(\mathrm{M})$ *for every* $t\in(0,1)$ *and for every* $\varepsilon\in(0,1/2)$ *it holds*

$$\int_\varepsilon^{1-\varepsilon} \||\nabla v_t|_{\mathsf{HS}}\|^2_{L^2(\mathrm{M})}\,\mathrm{d}\mathfrak{m} + \sup_{t\in(\varepsilon,1-\varepsilon)}\Big(\|v_t\|_{L^\infty(\mathrm{M})} + \|\mathrm{div}(v_t)\|_{L^\infty}\Big) < \infty$$

Actually this result as presented in [43] has a quite different statement: we see how we can pass from the original form to the one introduced here. The vector fields v_t are obtained as gradients of solutions η_t of a double obstacle problem, the obstacles being given by appropriate 'forward' and 'backward' Kantorovich potentials. This grants that (i) holds. Then (ii) is a general property of 'optimal' lifting of solutions of the continuity equation (see e.g. [36]). The estimates in (iii) are the main gain from [43]: the Laplacian comparison for the squared distance and the Lewy-Stampacchia inequality grant the claimed uniform control on $\mathrm{div}(v_t)=\Delta\eta_t$. With a cut-off procedure based on the fact that μ_0,μ_1 are assumed to have bounded support, one can show that the η_t's can be chosen to also have uniformly bounded support: this and the L^∞-bound on the Laplacian implies an L^2-bound on the Laplacian itself, so that from (1.4.4.9) we get the L^2-control on $|\nabla v_t|_{\mathsf{HS}} = |\mathrm{Hess}(\eta_t)|_{\mathsf{HS}}$. Finally, in [43] it has been proved that the η_t's are Lipschitz and, although not explicitly mentioned, keeping track of the various constants involved one can see that it is provided a uniform control on the Lipschitz constant for $t\in(\varepsilon,1-\varepsilon)$, which in turn implies the desired L^∞ control on $|v_t|$.

Thanks to Proposition 3.4.5 we can now prove the following crucial statement:

Proposition 3.4.6. *With the same notations and assumptions as in Section 3.4.1 the following holds. For every $x, y \in \mathrm{M}$ there exists $\underline{a} \in \mathbb{R}^N$ such that*

$$\mathsf{T}(x, \underline{a}) = y \qquad \textit{and} \qquad |\underline{a}| \leq \mathsf{d}(x, y).$$

proof Fix $y \in \mathrm{M}$, $R > 0$, define

$$\mu_0 := \mathfrak{m}(B_R(y))^{-1}\mathfrak{m}|_{B_R(y)} \qquad \text{and} \qquad \mu_1 := \delta_y$$

and let (μ_t) be the unique W_2-geodesic connecting μ_0 to μ_1 and $\boldsymbol{\pi}$ its lifting. Also, fix $\varepsilon \in (0, 1/2)$. Then we know from [44] that the W_2-geodesic $t \mapsto \mu_t^\varepsilon := \mu_{\varepsilon+(1-2\varepsilon)t}$ satisfies the assumptions of Proposition 3.4.5 and that its lifting $\boldsymbol{\pi}^\varepsilon$ is given by

$$\boldsymbol{\pi}^\varepsilon = (\mathrm{Restr}_\varepsilon^{1-\varepsilon})_*\boldsymbol{\pi},$$

where $\mathrm{Restr}_{t_0}^{t_1} : C([0,1], \mathrm{M}) \to C([0,1], \mathrm{M})$ is given by

$$\mathrm{Restr}_{t_0}^{t_1}(\gamma)_t := \gamma_{(1-t)t_0+tt_1} \qquad \forall \gamma \in C([0,1], \mathrm{M}).$$

Up to pass to a further restriction, Proposition 3.4.5 grants the existence of vector fields (v_t^ε) satisfying $(i), (ii), (iii)$ in the statement. In particular, by (iii) we know that the assumptions of Theorem 2.5.5 are satisfied so that there exists the Regular Lagrangian Flow $(\mathrm{Fl}_s^{(v_t^\varepsilon)})$ of (v_t^ε).

The representation formula for the solutions of the continuity equation given in Theorem 2.5.6 gives

$$\mu_s^\varepsilon = (\mathrm{Fl}_s^{(v_t^\varepsilon)})_*\mu_0^\varepsilon, \qquad \forall s \in [0,1]. \tag{3.4.3.2}$$

Thus letting $A_{i,t}^\varepsilon$ be defined by (3.4.2.9) for the vector fields (v_t^ε), from (3.4.2.14) we deduce that

$$\mathsf{T}(x, \underline{A}_1^\varepsilon(x)) = \mathrm{Fl}_1^{(v_t^\varepsilon)}(x) \in \mathrm{supp}(\mu_1^\varepsilon) \subset B_{\varepsilon R}(y) \qquad \mu_0^\varepsilon - a.e.\ x. \tag{3.4.3.3}$$

Now notice that $\boldsymbol{\pi}$ is concentrated on constant speed geodesics of length bounded above by R, hence the same holds for $\boldsymbol{\pi}^\varepsilon$, so that from (3.4.3.1) and (3.4.3.2) we deduce that

$$|v_s^\varepsilon| \circ \mathrm{Fl}_s^{(v_t^\varepsilon)} \leq R \qquad \mu_0^\varepsilon - a.e.. \tag{3.4.3.4}$$

Therefore using the trivial inequality

$$|\underline{A}_1^\varepsilon|^2 = \sum_{i=1}^N |\underline{A}_{i,1}^\varepsilon|^2 \leq \sum_{i=1}^N \int_0^1 |\langle v_s^\varepsilon, X_i\rangle|^2 \circ \mathrm{Fl}_s^{(v_t^\varepsilon)}\,\mathrm{d}s = \int_0^1 |v_s^\varepsilon|^2 \circ \mathrm{Fl}_s^{(v_t^\varepsilon)}\,\mathrm{d}s \overset{(3.4.3.4)}{\leq} R^2$$

valid μ_0^ε-a.e. in conjunction with (3.4.3.3) we deduce that for $\mu_0^\varepsilon = \mu_\varepsilon$-a.e. x

$$\text{there exists } \underline{a} \in \mathbb{R}^N \text{ with } |\underline{a}| \leq R \text{ such that } \mathsf{d}(\mathsf{T}(x, \underline{a}), y) \leq \varepsilon R \tag{3.4.3.5}$$

and an argument based on the continuity of T and the compactness of $B_R(0) \subset \mathbb{R}^N$ yields that the same holds for any $x \in \mathrm{supp}(\mu_\varepsilon)$.

Now notice that simple considerations about the structure of W_2-geodesics grant that the Hausdorff distance between $\mathrm{supp}(\mu_0) = B_R(y)$ and $\mathrm{supp}(\mu_\varepsilon)$ is bounded above by εR, thus for $x \in B_R(y)$ there is a sequence $n \mapsto x_n \in \mathrm{supp}(\mu_{1/n})$ converging to x. Let $\underline{a}_n$ be given by (3.4.3.5) for $x := x_n$ and $\varepsilon := \frac{1}{n}$: by the uniform bound $|\underline{a}_n| \leq R$ and up to pass to a non-relabeled subsequence we can assume that $\underline{a}_n \to \underline{a}$ for some $\underline{a} \in \mathbb{R}^N$ with $|\underline{a}| \leq R$. Passing to the limit in

$$\mathsf{d}\big(\mathsf{T}(x_n, \underline{a}_n), y\big) \leq \frac{R}{n}$$

using the continuity of T we conclude that $\mathsf{T}(x, \underline{a}) = y$. By the arbitrariness of $x \in B_R(y)$ and of $R > 0$ the proof is completed. $\square$

Let us now fix a point $\bar{x} \in \mathrm{M}$ and denote by $\mathbb{G} \subset \mathbb{R}^N$ its stabilizer, i.e.

$$\mathbb{G} := \Big\{ \underline{a} \in \mathbb{R}^N \ : \ \mathsf{T}(\bar{x}, \underline{a}) = \bar{x} \Big\}. \tag{3.4.3.6}$$

Notice that the last proposition (and the commutativity of $\mathbb{R}^N$) grants that the stabilizer does not depend on the choice of the particular point $\bar{x}$; moreover $\mathbb{G}$ is a subgroup of $\mathbb{R}^N$ which, by the continuity of T, is closed.

Proposition 3.4.7. *With the same notations and assumptions as in Section 3.4.1 the following holds. The subgroup $\mathbb{G}$ of $\mathbb{R}^N$ defined in* (3.4.3.6) *is discrete.*

proof We argue by contradiction. If it is not discrete, being closed it must contain a line so that for some $\underline{a} = (a_1, \dots, a_N) \neq 0$ in $\mathbb{R}^N$ we have $t\underline{a} \in \mathbb{G}$ for every $t \in \mathbb{R}$. Put $X := \sum_{i=1}^N a_i X_i$ and notice that X is not identically 0 and harmonic, so that its Regular Lagrangian Flow $(\mathrm{Fl}_t^{(X)})$ consists of measure preserving isometries of M such that for $\mathfrak{m}$-a.e. x the curve $t \mapsto \mathrm{Fl}_t^{(X)}(x)$ has constant positive speed. In particular, for $\mathfrak{m}$-a.e. x such curve is not constant.

On the other hand, the very definition of T yields

$$\mathsf{T}(x, t\underline{a}) = \mathrm{Fl}_t^{(X)}(x) \qquad \forall x \in \mathrm{M},\ t \in \mathbb{R}$$

and by assumption the left hand side is equal to x for every t: this gives the desired contradiction and the conclusion. □

The quotient space $\mathbb{R}^N/\mathbb{G}$ is equipped with the only Riemannian metric letting the quotient map be a Riemannian submersion. The distance induced by this metric is

$$\mathsf{d}_{\mathbb{R}^N/\mathbb{G}}\big([\underline{a}], [\underline{b}]\big) = \min_{\substack{\underline{a}' : [\underline{a}'] = [\underline{a}] \\ \underline{b}' : [\underline{b}'] = [\underline{b}]}} |\underline{a}' - \underline{b}'|. \tag{3.4.3.7}$$

Also, $\mathbb{R}^N/\mathbb{G}$ comes with a canonical, up to multiplication with a positive constant, reference measure $\mathfrak{m}_{\mathbb{R}^N/\mathbb{G}}$: the Haar measure, which also coincides with the volume measure induced by the metric.

In particular from the very definition we see that the map T passes to the quotient and induces a map $\tilde{\mathsf{T}} : \mathbb{R}^N/\mathbb{G} \to \mathrm{M}$ via the formula:

$$\tilde{\mathsf{T}}([\underline{a}]) := \mathsf{T}(\bar{x}, \underline{a}).$$

With this said, we can now conclude the proof of our main result:

Theorem 3.4.8. *With the same notations and assumptions as in Section 3.4.1 the following holds.*

i) The subgroup $\mathbb{G}$ of $\mathbb{R}^N$ defined in (3.4.3.6) *is isomorphic to $\mathbb{Z}^N$, so that the quotient space $\mathbb{R}^N/\mathbb{G}$ is a flat torus $\mathbb{T}^N$.*

ii) The induced quotient map $\tilde{\mathsf{T}} : \mathbb{T}^N \to \mathrm{M}$ is an isometry such that $\tilde{\mathsf{T}}_* \mathfrak{m}_{\mathbb{T}^N} = c\mathfrak{m}$ for some $c > 0$.

proof We subdivided the proof in the following three steps:

$\tilde{\mathsf{T}}$ is an isometry: From (3.4.1.8) and the definition (3.4.3.7) we get

$$\mathsf{d}\big(\tilde{\mathsf{T}}([\underline{a}]), \tilde{\mathsf{T}}([\underline{b}])\big) \leq \mathsf{d}_{\mathbb{R}^N/\mathbb{G}}\big([\underline{a}], [\underline{b}]\big) \qquad \forall [\underline{a}], [\underline{b}] \in \mathbb{R}^N/\mathbb{G}. \tag{3.4.3.8}$$

Now let $x, y \in \mathrm{M}$ and apply twice Proposition 3.4.6 to find $\underline{a} \in \mathbb{R}^N$ such that $\mathsf{T}(x, \underline{a}) = y$ and $|\underline{a}| \leq \mathsf{d}(x, y)$ and $\underline{b} \in \mathbb{R}^N$ such that $\mathsf{T}(\bar{x}, \underline{b}) = x$. Then we have

$$\mathsf{d}_{\mathbb{R}^N/\mathbb{G}}\big([\underline{b}], [\underline{a} + \underline{b}]\big) \leq |\underline{a}| \leq \mathsf{d}(x, y),$$

and since by construction and from (3.4.1.6) we have $\tilde{\mathsf{T}}([\underline{b}]) = x$ and $\tilde{\mathsf{T}}([\underline{a}+\underline{b}]) = y$, this inequality together with (3.4.3.8) shows that $\tilde{\mathsf{T}} : \mathbb{R}^N/\mathbb{G} \to \mathrm{M}$ is an isometry.

Up to a multiplicative constant, $\tilde{\mathsf{T}}$ is measure preserving: Being an isometry, $\tilde{\mathsf{T}}$ is invertible: denote by $\mathsf{S} : \mathrm{M} \to \mathbb{R}^N/\mathbb{G}$ its inverse and put $\mu := \mathsf{S}_*\mathfrak{m}$. For $\underline{a} = (a_1, \ldots, a_N) \in \mathbb{R}^N$ let $X_{\underline{a}} := \sum_{i=1}^N a_i X_i$ and notice that (3.4.1.6) reads as $\mathsf{T}(\bar{x}, \underline{b} + \underline{a}) = \mathrm{Fl}_1^{(X_{\underline{a}})}(\mathsf{T}(\bar{x}, \underline{b}))$ for every $\underline{a}, \underline{b} \in \mathbb{R}^N$. Passing to the quotient we obtain

$$\tilde{\mathsf{T}}([\underline{b}] + [\underline{a}]) = \mathrm{Fl}_1^{(X_{\underline{a}})}(\tilde{\mathsf{T}}([\underline{b}])) \qquad \forall \underline{a}, \underline{b} \in \mathbb{R}^N, \tag{3.4.3.9}$$

hence letting $\tau^{[\underline{a}]} : \mathbb{R}^N/\mathbb{G} \to \mathbb{R}^N/\mathbb{G}$ be the translation by $[\underline{a}]$ defined by $\tau^{[\underline{a}]}([\underline{b}]) := [\underline{b}] + [\underline{a}]$ we can rewrite (3.4.3.9) as

$$\tau^{[\underline{a}]} \circ \mathsf{S} = \mathsf{S} \circ \mathrm{Fl}_1^{(X_{\underline{a}})} \qquad \forall \underline{a} \in \mathbb{R}^N. \tag{3.4.3.10}$$

Therefore we have

$$\tau^{[\underline{a}]}_* \mu = \tau^{[\underline{a}]}_* \mathsf{S}_*\mathfrak{m} \overset{(3.4.3.10)}{=} \mathsf{S}_*(\mathrm{Fl}_1^{(X_{\underline{a}})})_*\mathfrak{m} \overset{(3.3.0.3)}{=} \mathsf{S}_*\mathfrak{m} = \mu \qquad \forall \underline{a} \in \mathbb{R}^N.$$

This shows that μ is translation invariant and thus a multiple of the Haar measure $\mathfrak{m}_{\mathbb{R}^N/\mathbb{G}}$.

The quotient space $\mathbb{R}^N/\mathbb{G}$ is a flat torus $\mathbb{T}^N$: What we just proved and (3.4.1.1) ensure that $\mathfrak{m}_{\mathbb{R}^N/\mathbb{G}}$ is a finite measure. Now recall that, as it is well known and trivial to prove, discrete subgroups of $\mathbb{R}^N$ are isomorphic to $\mathbb{Z}^n$ for some $n \leq N$ and that $\mathbb{R}^N/\mathbb{Z}^n$ has finite volume if and only if $n = N$. Being $\mathbb{G}$ discrete (Proposition 3.4.7), the thesis follows. $\square$

CHAPTER 4

Maximum principle on RCD spaces

4.1 Overview of the chapter

In this chapter we see a direct proof of the strong maximum principle on $\mathsf{RCD}^*(K,N)$ spaces, based just on the estimates for the Laplacian of the squared distance. We want to prove that if Ω an open and connected subset of M and $f \in W^{1,2}(\Omega) \cap C(\bar\Omega)$ is a subharmonic function on it such that for some $\bar x \in \Omega$ it holds $f(\bar x) = \max_{\bar\Omega} f$, then f is constant.

The first step consists in localizing the definition of Sobolev space over an open subset of the space (Definition 4.2.1) and this is possible just using the locality of the minimal weak upper gradient. We pass then to the definition of subharmonic and superharmonic functions, which is given in terms of the Dirichlet integral. We point out that in $\mathsf{RCD}(K,\infty)$ spaces, where the Sobolev-to-Lipschitz property holds, the proof of the weak maximum principle can be directly deduced from this definition.

As for the proof of the strong maximum principle we first need to characterize the subharmonic functions as those functions for which the measure valued Laplacian is a non negative measure (Theorem 4.2.4). Hence we recall that on $\mathsf{RCD}^*(K,N)$ spaces for N finite also the squared distance function $\mathsf{d}^2_{x_0}(\cdot)$ from a given point $x_0 \in$ M has a measure valued Laplacian, and the following bound holds

$$\boldsymbol{\Delta}\mathsf{d}^2_{x_0}(x) \le \ell_{K,N}(\mathsf{d}_{x_0})\,\mathfrak{m},$$

where $\ell_{K,N}\colon [0,+\infty) \to [0,+\infty)$ is a continuous function which depends only on K and N.

Finally the other key ingredient we shall need is a geometric property of RCD spaces (Lemma 4.4.1), which ensures that given $C \subset$ M a closed set, then for $\mathfrak{m}$ every point $x \notin C$ there exists a unique point $y \in C$ which realizes the distance of x from C. In the Euclidean setting this is easy to prove thanks to the strict convexity of balls, but in general metric spaces the same property can fail, even in presence of a (non-Riemannian) curvature-dimension condition, see Remark 4.4.2. In our situation this can be proved using the existence of optimal transport maps proved in [44], see Lemma 4.4.1.

4.2 Notation and preliminary results

Recall that in Section 1.3.1 we have introduced the notion of minimal weak upper gradient and in (1.3.1.4) we have seen that it is a local object, i.e.:

$$|\mathrm{D}f| = |\mathrm{D}g| \qquad \mathfrak{m}-a.e. \text{ on } \{f=g\} \qquad \forall f,g \in W^{1,2}(\mathrm{M}). \tag{4.2.0.1}$$

Then the notion of Sobolev space over an open set can be easily given:

Definition 4.2.1 (Sobolev space on an open subset of M)**.** *Let* $(\mathrm{M}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and let* $\Omega \subset \mathrm{M}$ *open. Then we define*

$$W^{1,2}_{\mathsf{loc}}(\Omega) := \{f \in L^2_{\mathsf{loc}}(\Omega) : \textit{for every } x \in \Omega \textit{ there exists } U \subset \Omega \textit{ neighbourhood of } x \\ \textit{and there exists } f_U \in W^{1,2}_{\mathsf{loc}}(\mathrm{M}) \textit{ such that } f_{|_U} = f_U\}.$$

For $f \in W^{1,2}_{\mathsf{loc}}(\Omega)$ *the function* $|\mathrm{D}f| \in L^2_{\mathsf{loc}}(\Omega)$ *is defined as*

$$|\mathrm{D}f| := |\mathrm{D}f_U| \quad \mathfrak{m} - \textit{a.e. on } U,$$

where $|\mathrm{D}f_U|$ *is the minimal weak upper gradient of* f_U *and the locality of this object ensures that* $|\mathrm{D}f|$ *is well defined.*

Then we set

$$W^{1,2}(\Omega) := \{f \in W^{1,2}_{\mathsf{loc}}(\Omega) : f, |\mathrm{D}f| \in L^2(\Omega)\}.$$

The definition of (sub/super)-harmonic functions can be given in terms of minimizers of the Dirichlet integral (see [22] for a more detailed discussion on the topic):

Definition 4.2.2 (Subharmonic/Superharmonic/Harmonic functions)**.** *Let* $(\mathrm{M}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and* Ω *be an open subset in* M*. We say that* f *is subharmonic (resp. superharmonic) in* Ω *if* $f \in W^{1,2}(\Omega)$ *and for any* $g \in W^{1,2}(\Omega)$*,* $g \leq 0$ *(resp.* $g \geq 0$*) with* $\operatorname{supp} g \subset \Omega$*, it holds*

$$\frac{1}{2}\int_\Omega |\mathrm{D}f|^2 \,\mathrm{d}\mathfrak{m} \leq \frac{1}{2}\int_\Omega |\mathrm{D}(f+g)|^2 \,\mathrm{d}\mathfrak{m}. \tag{4.2.0.2}$$

We say that f *is harmonic in* Ω *if it is both subharmonic and superharmonic.*

Instead in Section 1.4.3 we have defined the measure valued Laplacian for a Sobolev function, that is a key ingredient in the proof of the strong maximum principle. For that purpose we restrict the attention to proper (namely, closed bounded sets are compact) and infinitesimally Hilbertian (see Definition 1.3.24) spaces and also in this case we localize the definition in the following way:

Definition 4.2.3 (Measure valued Laplacian). *Let* $(\mathrm{M}, \mathsf{d}, \mathfrak{m})$ *be proper and infinitesimally Hilbertian,* $\Omega \subset \mathrm{M}$ *open and* $f \in W^{1,2}(\Omega)$*. We say that* f *has a measure valued Laplacian in* Ω*, and write* $f \in D(\mathbf{\Delta}, \Omega)$*, provided there exists a Radon measure, that we denote by* $\mathbf{\Delta} f_{|_\Omega}$*, such that for every* $g \colon \mathrm{M} \to \mathbb{R}$ *Lipschitz with support compact and contained in* Ω *it holds*

$$\int g \,\mathrm{d}\mathbf{\Delta} f_{|_\Omega} = -\int \langle \nabla f, \nabla g \rangle \,\mathrm{d}\mathfrak{m}. \tag{4.2.0.3}$$

If $\Omega = \mathrm{M}$ *we write* $f \in D(\mathbf{\Delta})$ *and* $\mathbf{\Delta} f$*.*

In the same way as in the smooth case, it turns out that being subharmonic is equivalent to having non-negative Laplacian. This topic has been investigated in [38] and [42], here we report the proof of this fact because in [42] it has been assumed the presence of a Poincaré inequality, while working on proper infinitesimally Hilbertian spaces allows to easily remove such assumption.

Theorem 4.2.4. *Let* $(\mathrm{M}, \mathsf{d}, \mathfrak{m})$ *be a proper infinitesimally Hilbertian space,* $\Omega \subset \mathrm{M}$ *open and* $f \in W^{1,2}(\Omega)$*.*

Then f *is subharmonic (resp. superharmonic, resp. harmonic) if and only if* $f \in D(\mathbf{\Delta}, \Omega)$ *with* $\mathbf{\Delta} f_{|_\Omega} \geq 0$ *(resp.* $\mathbf{\Delta} f_{|_\Omega} \leq 0$*, resp.* $\mathbf{\Delta} f_{|_\Omega} = 0$*).*

proof
Only if: Let $\mathrm{LIP}_c(\Omega) \subset W^{1,2}(\Omega)$ be the space of Lipschitz functions with support compact and contained in Ω. For $g \in \mathrm{LIP}_c(\Omega)$ non-positive and $\varepsilon > 0$ apply (4.2.0.2) with εg in place of g to deduce

$$\int_\Omega |\mathrm{D}(f+\varepsilon g)|^2 - |\mathrm{D}f|^2 \,\mathrm{d}\mathfrak{m} \geq 0$$

and dividing by ε and letting $\varepsilon \downarrow 0$ we conclude

$$\int_\Omega \langle \nabla f, \nabla g \rangle \,\mathrm{d}\mathfrak{m} \geq 0.$$

In other words, the linear functional $\mathrm{LIP}_c(\Omega) \ni g \mapsto -\int_\Omega \langle \nabla f, \nabla g \rangle \,\mathrm{d}\mathfrak{m}$ is positive. It is then well known, see e.g. [23, Theorem 7.11.3], that the monotone extension of such functional to the space of continuous and compactly supported functions on Ω is uniquely represented by integration w.r.t. a non-negative measure, which is the claim.
If: Recall from [8] that on general metric measure spaces Lipschitz functions are dense in energy in $W^{1,2}$; since infinitesimally Hilbertianity implies uniform convexity of $W^{1,2}$, we see that in our case they are dense in the $W^{1,2}$-norm. Then by truncation and cut-off argument we easily see that

$$\big\{g \in \mathrm{LIP}_c(\Omega) : g \leq 0\big\} \quad \text{is } W^{1,2}\text{−dense in} \quad \big\{g \in W^{1,2}(\Omega) : \ g \leq 0 \ \operatorname{supp}(g) \subset \Omega\big\}. \tag{4.2.0.4}$$

Now notice that the convexity of $g \mapsto \frac{1}{2}\int_\Omega |\mathrm{D}g|^2 \,\mathrm{d}\mathfrak{m}$ grants that for any $g \in W^{1,2}(\Omega)$ it holds

$$|\mathrm{D}(f+g)|^2 - |\mathrm{D}f|^2 \geq \lim_{\varepsilon \downarrow 0} \frac{|\mathrm{D}(f+\varepsilon g)|^2 - |\mathrm{D}f|^2}{\varepsilon} = 2\langle \nabla f, \nabla g \rangle$$

and thus from the assumption $\boldsymbol{\Delta} f_{|\Omega} \geq 0$ we deduce that

$$\int_\Omega |\mathrm{D}(f+g)|^2 - |\mathrm{D}f|^2 \,\mathrm{d}\mathfrak{m} \geq 0 \tag{4.2.0.5}$$

for every $g \in \mathrm{LIP}_c(\Omega)$ non-positive. Taking (4.2.0.4) into account we see that (4.2.0.5) also holds for any $g \in W^{1,2}(\Omega)$ non-negative with $\operatorname{supp}(g) \subset \Omega$, which is the thesis. □

For $x \in \mathrm{M}$ we write d_x for the function $y \mapsto \mathsf{d}(x,y)$. We shall need the following two properties of the squared distance function valid on $\mathsf{RCD}^*(K,N)$ spaces, $N < \infty$:

$$\mathsf{d}^2_{x_0} \in W^{1,2}_{\mathsf{loc}}(\mathrm{M}) \quad \text{and} \quad |\mathrm{D}(\mathsf{d}^2_{x_0})|^2 = 2\mathsf{d}^2_{x_0} \quad \mathfrak{m}\text{-a.e.}, \tag{4.2.0.6}$$

$$\mathsf{d}^2_{x_0} \in D(\boldsymbol{\Delta}) \quad \text{and} \quad \boldsymbol{\Delta}\mathsf{d}^2_{x_0}(x) \leq \ell_{K,N}(\mathsf{d}_{x_0})\mathfrak{m}, \tag{4.2.0.7}$$

where $\ell_{K,N} : [0,+\infty) \to [0,+\infty)$ is some continuous function depending only on K, N defined by

$$\ell_{K,N}(\theta) := \begin{cases} \left(1 + \theta\sqrt{K(N-1)}\mathrm{cotan}\left(\theta\sqrt{\dfrac{K}{N-1}}\right)\right), & \text{if} \quad K > 0, \\ N, & \text{if} \quad K = 0, \\ \left(1 + \theta\sqrt{-K(N-1)}\mathrm{cotan}\left(\theta\sqrt{\dfrac{-K}{N-1}}\right)\right), & \text{if} \quad K < 0. \end{cases}$$

As for property (4.2.0.6) we recall that can be seen as a consequence of Cheeger's work [27]. Indeed $\mathsf{CD}(K,N)$ spaces are doubling (as proved in [68]) and support a 1-2 weak Poincaré inequality (see [60]). Moreover, being M geodesic, the local Lipschitz constant of d_x is identically 1.

The Laplacian comparison estimate (4.2.0.7) is one of the results in [38], where actually such inequality has been obtained in its sharp form, but for our purposes the above formulation is sufficient.

Finally we state a useful result proved in [44]:

Theorem 4.2.5 (Exponentiation and optimal maps)**.** *Let $K \in \mathbb{R}$, $N \in [1,\infty)$, $(\mathrm{M},\mathsf{d},\mathfrak{m})$ an $\mathsf{RCD}^*(K,N)$ space, φ a c-concave function and $\Omega \subset \mathrm{M}$ the interior of $\{\varphi > -\infty\}$. Then for $\mathfrak{m}$-a.e. $x \in \Omega$ there exists a unique geodesic γ with $\gamma_0 = x$ and $\gamma_1 \in \partial^c\varphi(x)$.*

4.3 Weak maximum principle on $\mathsf{RCD}(K,\infty)$ spaces

On $\mathsf{RCD}(K,\infty)$ spaces, the weak maximum principle can be deduced directly from the definition of subharmonic function and the Sobolev-to-Lipschitz property in Theorem 1.4.2.

Theorem 4.3.1 (Weak Maximum Principle)**.** *Let $(\mathrm{M},\mathsf{d},\mathfrak{m})$ be an $\mathsf{RCD}(K,\infty)$ space, $\Omega \subset \mathrm{M}$ open and let $f \in W^{1,2}(\Omega) \cap C(\bar{\Omega})$ be subharmonic. Then*

$$\sup_{\Omega} f \le \sup_{\partial\Omega} f, \tag{4.3.0.1}$$

to be intended as 'f is constant' in the case $\Omega = \mathrm{M}$.

proof We argue by contradiction. If (4.3.0.1) does not hold, regardless of whether Ω coincides with M or not, we can find $c < \sup_\Omega f$ such that the function

$$\tilde{f} := \min\{c, f\}$$

agrees with f on $\partial\Omega$. The locality of the differential grants that

$$|\mathrm{D}\tilde{f}| = \chi_{\{f<c\}}|\mathrm{D}f| \tag{4.3.0.2}$$

and from the assumption that f is subharmonic and the fact that $\tilde{f} \le f$ we deduce that

$$\int_\Omega |\mathrm{D}f|^2 \,\mathrm{d}\mathfrak{m} \le \int_\Omega |\mathrm{D}\tilde{f}|^2 \,\mathrm{d}\mathfrak{m} \overset{(4.3.0.2)}{=} \int_{\{f<c\}\cap\Omega} |\mathrm{D}f|^2 \,\mathrm{d}\mathfrak{m},$$

which forces

$$|\mathrm{D}f| = 0 \ \mathfrak{m}\text{-a.e. on } \{f \ge c\}. \tag{4.3.0.3}$$

Now consider the function $g := \max\{c, \chi_\Omega f\}$, notice that our assumptions grant that $g \in C(\mathrm{M})$ and that the locality of the differential yields

$$|\mathrm{D}g| = \chi_{\Omega\cap\{f>c\}}|\mathrm{D}f| \overset{(4.3.0.3)}{=} 0. \tag{4.3.0.4}$$

Hence the Sobolev-to-Lipschitz property gives that g is constant, i.e. $f \le c$ on Ω. This contradicts our choice of c and gives the conclusion. □

We remark that in the finite-dimensional case one could conclude from (4.3.0.4) by using the Poincaré inequality in place of the Sobolev-to-Lipschitz property.

4.4 Strong Maximum Principle on $\mathsf{RCD}^*(K,N)$ spaces

We start proving the following geometric property of RCD spaces:

Lemma 4.4.1 (a.e. unique projection)**.** *Let $K \in \mathbb{R}$, $N \in [1,\infty)$, $(\mathrm{M},\mathsf{d},\mathfrak{m})$ an $\mathsf{RCD}^*(K,N)$ space and $C \subset \mathrm{M}$ a closed set. Then for $\mathfrak{m}$-a.e. $x \in \mathrm{M}$ there exists a unique $y \in C$ such that*

$$\mathsf{d}(x,y) = \min_{z\in C} \mathsf{d}(x,z). \tag{4.4.0.1}$$

proof Existence follows trivially from the fact that M is proper. For uniqueness define

$$\varphi(x) := \inf_{z\in C} \frac{\mathsf{d}^2(x,z)}{2} = \psi^c(x) \qquad \text{where} \qquad \psi(y) := \begin{cases} 0, & \text{if } y \in C, \\ -\infty, & \text{if } y \in \mathrm{M}\setminus C. \end{cases}$$

Since $\varphi^c = \psi^{cc} \geq \psi$, if $x \in \mathrm{M}$ and $y \in C$ are such that (4.4.0.1) holds, we have

$$\varphi(x) + \varphi^c(y) \geq \varphi(x) + \psi(y) \overset{(4.4.0.1)}{=} \frac{\mathsf{d}^2(x,y)}{2},$$

i.e. $y \in \partial^c\varphi(x)$. Conclude recalling that since φ is c-concave and real valued, Theorem 4.2.5 grants that for $\mathfrak{m}$-a.e. x there exists a unique $y \in \partial^c\varphi(x)$. □

Remark 4.4.2. The proof of this lemma, even if is very simple, relies on quite delicate properties of RCD spaces: notice indeed that the conclusion can fail on the more general $\mathsf{CD}(K,N)$ spaces. Consider for instance $\mathbb{R}^2$ equipped with the distance coming from the L^∞ norm and the Lebesgue measure $\mathcal{L}^2$. This is a $\mathsf{CD}(0,2)$ space, as shown in the last theorem in [69]. Then pick $C := \{(z_1,z_2) : z_1 \geq 0\}$ and notice that for every $(x_1,x_2) \in \mathbb{R}^2$ with $x_1 < 0$ there are uncountably many minimizers in (4.4.0.1).

What makes the proof work in the RCD case is the validity of the result in [44] which uses some forms of non-branching and lower Ricci bounds to deduce existence of optimal maps. This kind of argument appeared first in [34]. ■

We can now prove the main result of this section:

Theorem 4.4.3 (Strong Maximum Principle)**.** *Let $K \in \mathbb{R}$, $N \in [1,\infty)$ and $(\mathrm{M},\mathsf{d},\mathfrak{m})$ an $\mathsf{RCD}^*(K,N)$ space. Let $\Omega \subset \mathrm{M}$ be open and connected and let $f \in W^{1,2}(\Omega) \cap C(\bar\Omega)$ be subharmonic and such that for some $\bar x \in \Omega$ it holds $f(\bar x) = \max_{\bar\Omega} f$. Then f is constant.*

proof Put $m := \sup_\Omega f$, $C := \{x \in \bar\Omega : f(x) = m\}$ and define

$$\Omega' := \big\{x \in \Omega \setminus C \ : \ \mathsf{d}(x,C) < \mathsf{d}(x,\partial\Omega)\big\}.$$

By assumption we know that $C \cap \Omega \neq \emptyset$ and that Ω is connected, thus since C is closed, either $C \supset \Omega$, in which case we are done, or $\partial C \cap \Omega \neq \emptyset$, in which case $\Omega' \neq \emptyset$. We now show that such second case cannot occur, thus concluding the proof.

Assume by contradiction that $\Omega' \neq \emptyset$, notice that Ω' is open and thus $\mathfrak{m}(\Omega') > 0$. Hence by Lemma 4.4.1 we can find $x \in \Omega'$ and $y \in C$ such that (4.4.0.1) holds. Notice that the definition of Ω' grants that $y \in \Omega$, put $r := \mathsf{d}(x,y)$ and define

$$h(z) := e^{-A\mathsf{d}^2(z,x)} - e^{-Ar^2},$$

where $A \gg 1$ will be fixed later. By the chain rule for the measure-valued Laplacian (see [38]) we have that $h \in D(\mathbf{\Delta})$ with

$$\mathbf{\Delta} h = A^2 e^{-A\mathsf{d}_x^2}|\mathrm{D}\mathsf{d}_x^2|^2\,\mathfrak{m} - Ae^{-A\mathsf{d}_x^2}\mathbf{\Delta}\mathsf{d}_x^2 \overset{(4.2.0.6),(4.2.0.7)}{\geq} 2e^{-A\mathsf{d}_x^2}\big(A^2\mathsf{d}_x^2 - A\ell_{K,N}(\mathsf{d}_x)\big)\mathfrak{m}.$$

Hence we can, and will, choose A so big that $\boldsymbol{\Delta} h_{|B_{r/2}(y)} \geq 0$. Now let $r' < r/2$ be such that $B_{r'}(y) \subset \Omega$ and notice that for every $\varepsilon > 0$ the function $f_\varepsilon := f + \varepsilon h$ is subharmonic in $B_{r'}(y)$ and thus according to Theorem 4.3.1 we have

$$f_\varepsilon(y) \leq \sup_{\partial B_{r'}(y)} f_\varepsilon, \qquad \forall \varepsilon > 0. \tag{4.4.0.2}$$

Since $\{h < 0\} = \mathrm{M} \setminus \bar{B}_r(x)$ and $h(y) = 0$ we have

$$f_\varepsilon(y) > f_\varepsilon(z) \qquad \forall z \in \partial B_{r'}(y) \setminus \bar{B}_r(x),\ \forall \varepsilon > 0. \tag{4.4.0.3}$$

On the other hand, $\partial B_{r'}(y) \cap \bar{B}_r(x)$ is a compact set contained in $\Omega \setminus C$, hence by continuity and the definition of C we have

$$f(y) > \sup_{\partial B_{r'}(y) \cap \bar{B}_r(x)} f$$

and thus for $\varepsilon > 0$ sufficiently small we also have

$$f_\varepsilon(y) > \sup_{\partial B_{r'}(y) \cap \bar{B}_r(x)} f_\varepsilon.$$

This inequality, (4.4.0.3) and the continuity of f_ε contradict (4.4.0.2); the thesis follows. □

Remark 4.4.4. The proof uses the Laplacian comparison of the distance, its linearity and Lemma 4.4.1 only. Since the Laplacian comparison for the distance holds in the more general class of infinitesimally strictly convex MCP spaces (see [38]), taking Remark 4.4.2 into account we see that the strong maximum principle holds in the class of essentially non-branching and infinitesimally Hilbertian MCP spaces. ■

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